MTH101N Marking scheme

Qn. 1

a) $0\cdot 3 = \frac{3}{10}$ $0\cdot 33 = \frac{3}{10} + \frac{3}{10^2}$, $0\cdot 333 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3}$

If $a_k = \frac{3}{10^k}$ then the given seqn. is $s_n = \sum_{k=1}^{n} a_k$ — (3)

$\sum_{1}^{\infty} a_k = \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) = 1/3$

— (2)

b) $\sum a_n$ cvgs $\Rightarrow a_n \to 0$

$b_n = 1 - \frac{\sin a_n}{a_n} = a_n^3/3! - \frac{a_n^5}{5!} + \cdots$ (1)

$\frac{b_n}{a_n} = a_n^2/3! - a_n^4/5! + \cdots \to 0$ as $a_n \to 0$ (1)

(Since $\sum a_n$ cvgs, $\sum a_n^2$ cvgs) (2)

Thus $\sum b_n$ cvgs. (1)

(1)

c) $a_n \geq 2 \ \forall n$

$a_{n+1} - a_n = \frac{1}{a_n} + \frac{1}{a_n^2} - \left(\frac{1}{a_{n-1}} + \frac{1}{a_{n-1}^2}\right)$ (1)

$\Rightarrow |a_{n+1} - a_n| \leq \frac{1}{a_n a_{n-1}} |a_n - a_{n-1}| + \frac{1}{a_n^2 a_{n-1}^2} |a_n^2 - a_{n-1}^2|$ (2)

(1)

$= \frac{|a_n - a_{n-1}|}{a_n a_{n-1}} \left(1 + \frac{a_n + a_{n-1}}{a_n a_{n-1}}\right)$

$= \frac{|a_n - a_{n-1}|}{a_n a_{n-1}} \left(1 + \frac{1}{a_n} + \frac{1}{a_{n-1}}\right)$

$\leq \frac{1}{4} |a_n - a_{n-1}| \left(1 + \frac{1}{2} + \frac{1}{2}\right)$ (1)

$\leq \frac{1}{2} |a_n - a_{n-1}|$ (1)

Thus (a_n) is Cauchy.

MTH101N Marking scheme

Qn 2

a) $\dfrac{f(b)-f(a)}{b-a} = \dfrac{f(a)f(b)}{ab} \Rightarrow \dfrac{1}{f(b)} - \dfrac{1}{b} = \dfrac{1}{f(a)} - \dfrac{1}{a}$

Let $g(x) = \dfrac{1}{f(x)} - \dfrac{1}{x}$: g is well defined diff'le fn. (2)

$g(b) = g(a) \Rightarrow \exists\, x_0 \in (a,b)$ such that $g'(x_0) = 0$ (2)

$\Rightarrow x_0^2 f'(x_0) = f'(x_0)$ (2)

(1)

b) $f(x) = \dfrac{x^2}{x-1} = (x+1) + \dfrac{1}{x-1}$

(i) $y = x+1$, $x = 1$ are asymptotes

(2)

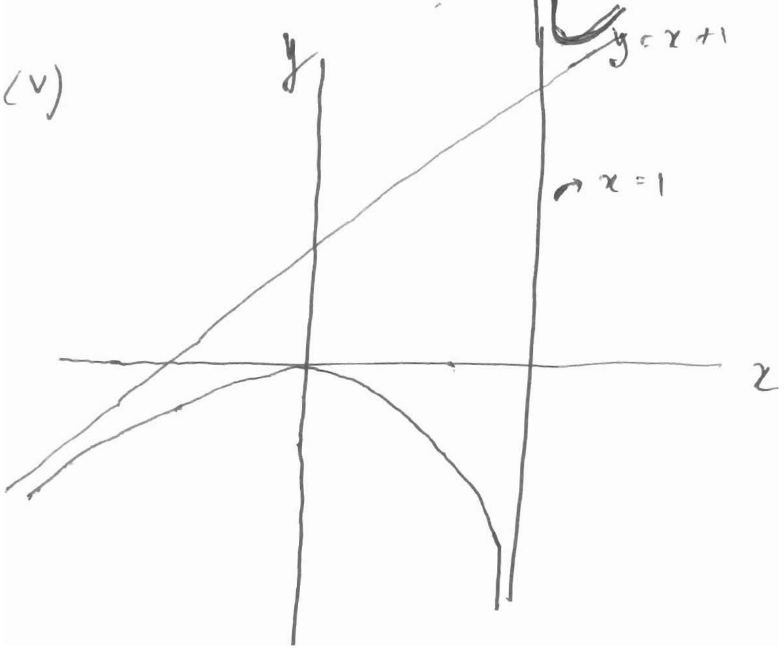
(ii) $f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ $\begin{cases} <0 & \text{on } (0,1)\cup(1,2] \quad \text{decreasing} \\ >0 & \text{on } (-\infty,0)\cup(2,\infty) \quad \text{increasing} \end{cases}$

(2)

(iii) $f''(x) = \dfrac{2}{(x-1)^3}$ $\begin{cases} <0 & \text{on } (-\infty,1) \quad \text{Concave} \\ >0 & \text{on } (1,\infty) \quad \text{Convex.} \end{cases}$

(iv) max at 0, min at 2 (2)

(v) (2)



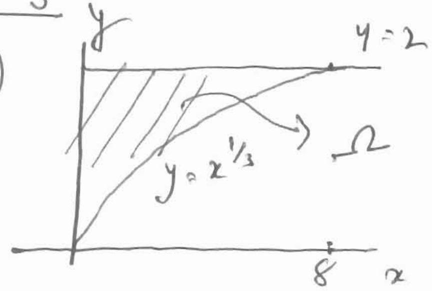
(2)

MTH101N

Marking scheme

Qn. 3

a)



$$\text{Area} = \iint_{\Omega} dx\, dy$$

(1)

$$\Omega = \{ 0 \le y \le 2 : 0 \le x \le y^3 \}$$

$$\text{hence Area} = \int_0^2 \left(\int_0^{y^3} dx \right) dy = 4$$

(2)

b) $\text{Area}(D) = \iint_D dx\, dy$

(2)

If $M(x,y) = -y$, $N(x,y) = 0$ then $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$ — (1)

hence $\text{Area}(D) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$

(Green's Thm) $= \oint_C M\, dx + N\, dy = \oint_c -y\, dx$

(2)

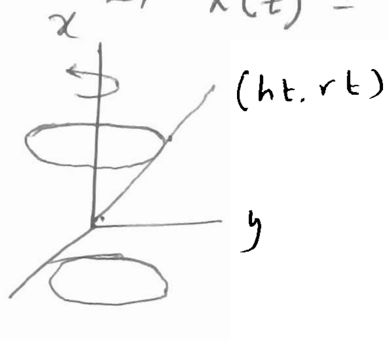
Similarly if $M(x,y) = 0$, $N(x,y) = x$, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$

$\Rightarrow \text{Area}(D) = \oint M\, dx + N\, dy = \oint_C x\, dy$ (1)

It also follows $\text{Area}(D) = \frac{1}{2} \oint_C -y\, dx + x\, dy$

(1)

c) Let $R(t) = (ht, rt)$ t $[0,1)$



Rotate $R(t)$ wrt. x-axis

(3)

parametrized the solid generated as

$x = ht$, $y = rt \cos\theta$, $z = rt \sin\theta$

$t \in [0,1)$, $\theta \in [0, 2\pi)$

(2)

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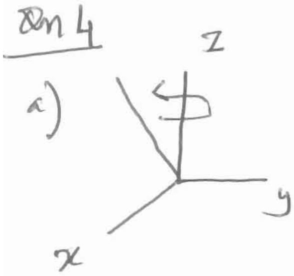
Qn 4

a) 

In the x-z plane consider the fn. $f(z) = z$ and rotate it around z-axis. (2)

The solid generated between $0 \le z \le 1$ has

$$\text{volume} = \int_0^1 \pi z^2 \, dz = \pi/3$$ (4)

[alt. $$\text{volume} = \int_0^1 \Big(\iint_{x^2+y^2 \le z} dx\,dy \Big) dz = \int_0^1 \pi z \, dz = \pi/3$$ (6)]

b) Find max. and min of $f(x,y,z) = (x-\frac{1}{2})^2 + (y-1/2)^2 + z^2$ subjected to $g(x,y,z) = x^2+y^2+2z^2 = 1$ (1)

By Lagrange's multipliers method

$$2(x_0 - 1/2) = 2\lambda x_0 \Rightarrow (1-\lambda)x_0 = 1/2$$

$$2(y_0 - 1/2) = 2\lambda y_0 \Rightarrow (1-\lambda)y_0 = 1/2$$

$$2z_0 = 4\lambda z_0 \Rightarrow \lambda = 1/2 \text{ or } z_0 = 0$$ (2)

if $\lambda = 1/2$, $x_0 = 1$, $y_0 = 1$ hence $(x_0, y_0, 2z_0)$ can not be the given surface, thus $z_0 = 0$ (1)

Since $x_0^2 + y_0^2 = 1 - 2z_0^2$ we have $(1-\lambda) = \pm \frac{1}{\sqrt{2}}$.

Thus max. distance to $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$

min. distance to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ (2)

c) To minimize $f(x,y,z) = (x-a)^2 + (y-b)^2 + (z-c)^2$ subjected to $g(x,y,z) = 0$. By Lagrange's multiplier's method

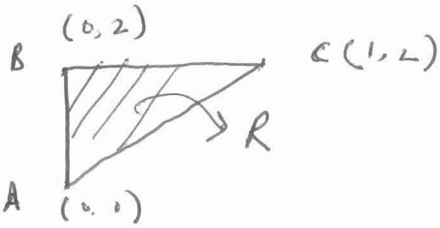
$$(x-a) = \lambda g_x \quad (y-b) = \lambda g_y \quad (z-c) = \lambda g_z$$ (3)

Thus the line $(x-a, y-b, z-c)$ is parallel to (g_x, g_y, g_z) which is normal to $g(x,y,z) = 0$

(2)

MTH101N Marking scheme

Qn. 5



a) $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$

In the interior of R

$f_x = 0 \Rightarrow 4x - 4 = 0 \Rightarrow x = 1$

$f_y = 0 \Rightarrow 2y - 4 = 0 \Rightarrow y = 2$

– no critical point in the interior of R (2)

Along AB : $f(0,y) = y^2 - 4y + 1 = s(y)$

$s'(y) = 0 \Rightarrow y = 2$ (1)

Along AC $f(x, 2x) = 6x^2 - 12x + 1 = h(x)$

$h'(x) = 0 \Rightarrow x = 1$ (1)

Along BC $f(x,2) = 2x^2 - 4x - 3 = k(x)$

$k'(x) = 0 \Rightarrow x = 1$ (1)

max. and min. can occur only at $(0,2)$, $(0,0)$, $(1,2)$

$f(0,0) = 1$ $f(1,2) = -5$ $f(0,2) = 3$

Max of f is 1 at $(0,0)$, min is -5 at $(1,2)$ (1)

b) $(Df)_{x_0}(u) = \nabla f(x_0)\cdot u$, $\|u\| = 1$ (2)

$= \|\nabla f(x_0)\| \cos\theta$ (2)

$\max (Df)_{x_0}(u) = \|\nabla f(x_0)\|$ at $\theta = 0$ (2)

c) $\nabla f(x,y,z) = \big(yz(1-x)(2-y)(3-z) - xyz(2-y)(3-z)\big)\hat{i}$

$+ \big(zx(1-x)(2-y)(3-z) - xyz(1-x)(3-z)\big)\hat{j}$

$+ \big(xy(1-x)(2-y)(3-z) - xyz(1-x)(2-y)\big)\hat{k}$ (2)

$\nabla f(\frac{1}{2}, 1, 1) = \frac{1}{4}k.$ (1)

By Qn 5(b) the mosquito should fly in the direction $-\frac{1}{4}k$ (2)

MTH101W Marking scheme

Qn 6

a) The boundary of the Solid $x^2 + y^2/4 + z^2/4 = 1$

unit normal $= \dfrac{1}{\sqrt{1+3x^2}}(2x,\ y/2,\ z/2)$ (1)

Let $F(x,y,z) = (x+3yz,\ 5zx+2y,\ x^2y^2+3z)$ (1)

Then given $\iint_S d\sigma = \iint_S F\cdot n\, d\sigma$ (2)

Div. Thm $= \iiint_V \operatorname{div} F\, dx\,dy\,dz$

$= \iiint 6\, dx\,dy\,dz = 6V$ (2)

b) Curl $F = \nabla \times F = (xz^2, -yz^2, 0)$ (2)

Unit normal to the Surface $= \dfrac{1}{\sqrt{1+12y^2}}(x,\ 4y,\ 0)$

Thus $\iint_S \dfrac{x^2z^2 - 4y^2z^2}{\sqrt{1+12y^2}} = \iint_S \text{Curl} F\cdot n\, d\sigma$ (2)

stoke's Thm $= \oint_c F\cdot dc$

At boundary curve $c(t)$, $z=-1$ and $z=1$ (2)

Hence $dc = \cdot\hat{i} + \cdot\hat{j} + 0\hat{k}$. Thus

$F\cdot dc = 0$. Hence $\iint_S = 0$ (1)

c) $\text{curl} F = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\hat{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\hat{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\hat{k}$ —(1)

$\operatorname{div}(\text{Curl} F) = \dfrac{\partial^2 R}{\partial x\partial y} - \dfrac{\partial^2 Q}{\partial x\partial z} + \dfrac{\partial^2 P}{\partial y\partial z} - \dfrac{\partial^2 R}{\partial y\partial x} + \dfrac{\partial^2 Q}{\partial z\partial x} - \dfrac{\partial^2 P}{\partial z\partial y}$ (1)

Using mixed derivative Theorem $= 0$ (1)