i Please write your Name, Roll Number and Section Number correctly on the answer booklet.

ii Attempt each question on a new page and attempt all parts of a question at the same place.

iii Attempt all questions (total 6 questions).

iv Please make a table on the front page indicating the question number and respective page number.

(1) (i) Let \( p \) be a quadratic polynomial. Answer True or False. Justify your answer.

(a) If \( (a_n) \) is a sequence converging to 0 then the sequence \( \{(-1)^n p(n) a_n\} \) converges to 0. 

(b) If \( a \) is a positive real number less than 1 then the sequence \( \{(-1)^n p(n) a^n\} \) converges to 0.

[4+4=8]

Solution:

(a) False. \hspace{2cm} [1]

Let \( p(x) = x^2 \) and \( a_n = 1/n \). Then \( (-1)^n p(n) a_n = (-1)^n n \) is divergent. \hspace{2cm} [3]

(b) True. \hspace{2cm} [1]

Let \( p(x) = a_0 + a_1 x + a_2 x^2 \). Then \( (-1)^n p(n) a_n = (-1)^n = (-1)^n (a_0 + a_1 n + a_2 n^2) x^n \). By Ratio Test, \( (-1)^n n^k x^n \) converges to 0 for any non-negative integer \( k \). Hence \( \{(-1)^n p(n) a^n\} \) also converges to 0. \hspace{2cm} [3]

(ii) Consider the function \( f(x) = \lim_{m \to \infty} \lim_{n \to \infty} (\cos(m! \pi x))^n \) for \( x \in \mathbb{R} \). Determine the points of continuity for the function \( f \).

Solution: If \( x \) is a rational, then for large \( m, m! x \) is an even integer. Hence \( \cos(m! \pi x) = 1 \). It follows that \( f(x) = 1 \) if \( x \) is rational. \hspace{2cm} [3]

If \( x \) is an irrational then \( |\cos(m! \pi x)| < 1 \), and hence \( \lim_{n \to \infty} (\cos(m! \pi x))^n = 0 \). It follows that \( f(x) = 0 \) if \( x \) is irrational. \hspace{2cm} [3]

Let \( x \) be irrational. Let \( x_n \) be a sequence of rationals such that \( x_n \to x \). Then \( f(x_n) = 1 \to 1 \) but \( f(x) = 0 \). Thus, \( f \) is discontinuous at \( x \).

Let \( x \) be rational. Let \( x_n \) be a sequence of irrationals such that \( x_n \to x \). Then \( f(x_n) = 0 \to 0 \) but \( f(x) = 1 \). Thus, \( f \) is discontinuous at \( x \).

(2) (i) Show that the improper integral \( \int_0^\infty e^{-t p^{-1}} dt \) converges for every positive number \( p \).

Solution: Note that \( \int_0^\infty e^{-t p^{-1}} dt = \int_0^1 e^{-t p^{-1}} dt + \int_1^\infty e^{-t p^{-1}} dt \). \hspace{2cm} [2]

By Limit Comparison Test with \( 1/t^2 \), \( \int_1^\infty e^{-t p^{-1}} dt \) converges for every positive number \( p \). \hspace{2cm} [2]

If \( p \geq 1 \), then \( \int_0^1 e^{-t p^{-1}} dt \) converges since \( e^{-t p^{-1}} \) is continuous on \( [0, 1] \). Hence \( \int_0^\infty e^{-t p^{-1}} dt \) converges for every \( p \geq 1 \). \hspace{2cm} [2]
If $0 < p \leq 1$, then $\int_0^1 e^{-tp^{-1}}dt$ converges by LCT with $1/t^{1-p}$. Hence $\int_0^\infty e^{-tp^{-1}}dt$ converges for every $p \geq 1$. 

Note: Give full marks in case all steps are there except first.

(ii) Let $n$ be a positive integer. Determine the values of $x$ for which the power series converges: (a) $\sum_{k=1}^\infty n^k x^k$, (b) $\sum_{k=1}^\infty k^n x^k$. 

Solution:

(a) $\sum_{k=1}^\infty n^k x^k = \sum_{k=1}^\infty (nx)^k$ converges if $|nx| < 1$, i.e., if $x \in (-1/n, 1/n)$.

(b) Let $a_n = k^n x^k$. Then $\frac{|a_{n+1}|}{|a_n|} \to |x|$. Hence, by Ratio Test, $\sum_{k=1}^\infty k^n x^k$ converges for $x \in (-1, 1)$.

(ii) For which base radius $r$ and height $h$ does a cone inscribed in a unit sphere in $\mathbb{R}^3$ have maximum volume.

Solution: Volume of the cone $f(r, h) = \frac{1}{3}\pi r^2 h$. We wish to maximise $f(r, h)$ with the constraint $g(r, h) = r^2 + (h - 1)^2 = 1$.

By the method of Lagrange multipliers:

$\nabla f = \lambda \nabla g \implies \left(\frac{2}{3}\pi r h, \frac{1}{3}\pi r^2\right) = \lambda (2r, 2(h - 1))$. 

Eliminating $\lambda$, we get $r^2 = 2h(h - 1)$. Since, $r^2 + (h - 1)^2 = 1$, we see that $h = \frac{4}{3}$ and $r = \frac{2\sqrt{2}}{3}$.

(iii) Let $f(x, y) = \frac{x\sqrt{|y|}}{\sqrt{x^2 + y^2}}$. Determine whether or not $\lim_{(x, y) \to (0, 0)} f(x, y)$ exists.

Solution: Since $\forall x, y$, $\sqrt{x^2} \leq \sqrt{x^2 + y^2}$, we observe that when $x \neq 0$

$$-\sqrt{|y|} \leq -\frac{|x|\sqrt{|y|}}{\sqrt{x^2}} \leq \frac{x\sqrt{|y|}}{\sqrt{x^2 + y^2}} \leq \frac{|x|\sqrt{|y|}}{\sqrt{x^2}} \leq \sqrt{|y|}$$.
If \( x = 0 \), then \( f(0, y) = 0 \sqrt{|y|} / \sqrt{y^2} = 0 \). \[1\]

Since \( \sqrt{\cdot} \) and \( |\cdot| \) are continuous functions by the Sandwich Theorem
\[ \lim_{(x,y)\to(0,0)} f(x,y) = 0. \] \[3\]

**Aliter:**
\[ |x| \leq \sqrt{x^2 + y^2} \]
\[ |f(x, y)| = |x| \sqrt{|y|} / \sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2} \sqrt{|y|} / \sqrt{x^2 + y^2} \leq \sqrt{|y|} \]
Hence as \( y \to 0 \), \( f(x, y) \to 0. \) \[2\]

(4) (i) Let \( C \) be a simple closed curve in \( \mathbb{R}^3 \). Let \( R : [a, b] \to \mathbb{R}^3 \) be a parametrization of \( C \) such that for all \( t \in [a, b] \), \( \| R'(t) \| = 1 \). The curvature at \( R(t) \) is given by \( \kappa(t) = \| R''(t) \| \) and the total curvature of \( C \) is defined by \( \int_C \kappa \, ds \).

Compute the total curvature of a circle \( C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = a^2, z = 0 \} \). \[5\]

**Solution:** Here \( R(s) = (a \cos(s/a), a \sin(s/a), 0) \), where \( 0 \leq s \leq 2\pi \). \[1\]
\( R'(s) = (-\sin(s/a), \cos(s/a), 0) \) is of unit length for all \( s \) and \( R''(s) = (\frac{-\cos(s/a)}{a}, -\frac{\sin(s/a)}{a}, 0) \) and \( \kappa(s) = \| R''(s) \| = \frac{1}{a} \). \[2\]
The total curvature is \( \int_C \kappa \, ds = \int_0^{2\pi} \frac{1}{a} \, ds = 2\pi \). \[2\]

(ii) Let \( D \) be the region in the first quadrant bounded by the hyperbolae \( xy = 1, xy = 9 \) and the line \( y = x \) and \( y = 4x \). Evaluate
\[ \int \int_D (\sqrt{y/x} + \sqrt{xy}) \, dx \, dy \]

**Solution:** Let \( xy = u^2 \) and \( \frac{y}{x} = v^2 \). \[1\]
Then \( 1 \leq u \leq 3 \) and \( 1 \leq v \leq 2 \) , \( x = u/v \) and \( y = uv \) \[2\]
The Jacobian \( J = \frac{2u}{v} \). \[1\]
Therefore by the change of variable formula
\[ \int \int_D (\sqrt{y/x} + \sqrt{xy}) \, dx \, dy = \int_1^2 \int_1^3 (u + v) \frac{2u}{v} \, du \, dv = 8 + \frac{52}{3} \ln 2. \] \[1+1\]

(iii) Determine whether or not the following vector fields are gradient vector fields. If yes, find the corresponding potential functions.
(a) \( F(x, y) = (xe^y, ye^x) \)
(b) \( F(x, y) = (-\frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2}) \), for \((x, y) \neq (0, 0)\). \[2+4\]

**Solution:**
(a) \( F(x, y) = (xe^y, ye^x) \implies P(x, y) = xe^y \text{ and } Q(x, y) = ye^x. \)
Since \( P_y = xe^y \text{ and } Q_x = ye^x \), \( P_y \neq Q_x \) implies that \( F \) is not a gradient vector field. \[2\]

(b) Here \( P_y = Q_x = x^2 - y^2 \left( \frac{x^2 + y^2}{2} \right)^2. \] \[1\]
If \( C \) is the unit circle about the origin then \( \int_C F.dR \neq 0 \) and thus \( F \) is not a gradient vector field. \[3\]

(5) (i) Evaluate \( \int \int \int_D x dV \) where \( D \) is the region enclosed by \( z = 0 \), \( z = x + y + 5 \), \( x^2 + y^2 = 4 \) and \( x^2 + y^2 = 9 \). \[5\]

**Solution:** Using cylindrical polar coordinates, we see that \( 0 \leq z \leq r \cos \theta + r \sin \theta + 5 \), \( 2 \leq r \leq 3 \) and \( 0 \leq \theta \leq 2\pi \).
\[
\int \int \int_D x dV = \int_0^2 \int_0^r \int_0^{r \cos \theta + r \sin \theta + 5} r \cos \theta r dz dr d\theta \]
\[
= \frac{65}{4}. \]

(ii) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(X) = \|X\| \).
(a) Evaluate \( f'(1,0) \).
(b) Determine whether \( f \) is differentiable at \( (0,0) \). \[2+4\]

**Solution:**
(a) \( f'(1,0) = \nabla(f)|_{(1,0)} = (1,0). \) \[2\]
(b) If \( f \) is differentiable at \( (0,0) \), then it is necessary for both \( f_x \) and \( f_y \) to exist at \( (0,0) \).
\[
\lim_{t \to 0} \frac{f((0,0) + t(1,0)) - f((0,0))}{t} = \lim_{t \to 0} \frac{|t|}{t} \text{ which does not exist} \]
Hence, \( f_x(0,0) \) does not exist and therefore \( f'(0,0) \) does not exist. \[2\]

(iii) Let \( S \) be the surface consisting of the top and the four sides of the cube whose vertices are \( \pm 1, \pm 1, \pm 1 \) oriented with outward normals. Let \( F(x,y,z) = x^2i + 2yj + e^zk \).
Evaluate the flux integral \( \int \int_S F.nd\sigma \). \[7\]

**Solution:** Let \( S_2 \) be the bottom of the cube oriented with a downward pointing normal. Then \( R = S \cup S_2 \) is the surface that bounds the cube \( C \).
\[
\text{div}(F) = 2x + 2 + e^z \]
By the divergence theorem
\[
\int \int_R F.nd\sigma = \int \int \int_C \text{div} F dV = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + 2 + e^z) dx dy dz = 16 - \frac{4}{e} + 4e. \]
Since \( S_2 \) is the portion of the plane \( z = -1 \), with \( -1 \leq x \leq 1 \) and \( -1 \leq y \leq 1 \), \( S_2 \) is the parametrised surface \( r(u,v) = \{(u,v,-1)| -1 \leq u \leq 1, -1 \leq v \leq 1 \}. \)
The unit normal to \( S_2, n = -(0,0,1). \) \[1\]
\[
\int \int_{S_2} F \cdot n \, d\sigma = - \int_{-1}^{1} \int_{-1}^{1} \left( u^2, 2v, \frac{1}{e} \right) \cdot (0, 0, 1) \, dx \, dy = -\frac{4}{e}. \tag{2}
\]

Hence \( \int \int_{S} F \cdot n \, d\sigma = 16 + 4e. \tag{1} \]

(6) (i) Find the volume of the solid region bounded by the cone \( z = \sqrt{3x^2 + 3y^2} \) and the sphere \( x^2 + y^2 + z^2 = 9. \) \( \tag{5} \]

**Solution:** The sphere and the cone will intersect in a circle of radius \( \frac{3}{2}. \)

Using spherical polar coordinates, we get the limits for \( \phi, \theta \) and \( \rho \) as:

\[
0 \leq \phi \leq \frac{\pi}{6}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \rho \leq 3. \tag{2}
\]

Hence Volume \( V = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{\frac{3}{2}} \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho = 18\pi \left(1 - \frac{3}{2} \right). \tag{2} \]

(ii) Let \( C \) be the quadrilateral with vertices \((1, 1), (1, 2), (2, 1)\) and \((2, 3)\) oriented clockwise. Let \( F \) be the vector field \( F(x, y) = (e^{x^4 + y^2}, xy + \sin(lny)). \) Evaluate the line integral of \( F \) over \( C. \) \( \tag{5} \]

**Solution:** By Green’s Theorem \( \int_{C} F \cdot dR = -\int_{D} \int -y \, dx \, dy \) where

\[
D = \{(x, y) \in \mathbb{R}^2 | 1 \leq y \leq x + 1, 1 \leq x \leq 2\}. \tag{3}\]

Therefore \( \int_{C} F \cdot dR = \frac{16}{6}. \tag{2} \]

(iii) Let \( r \geq 0 \) and \( S_r = \{ X \in \mathbb{R}^3 \mid \| X \| = r \} \) be the sphere of radius \( r \) centered at the origin. Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be a differentiable function and suppose \( \nabla f(X) = g(X)X, \forall X \in \mathbb{R}^3, \) where \( g : \mathbb{R}^3 \to \mathbb{R} \) is some function.

(a) Show that \( f \) is constant on each sphere \( S_r. \)

(b) Let \( h : (0, \infty) \to \mathbb{R} \) be a differentiable function given by

\[
h(\| X \|) = f(X) \text{ for all } X \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}. \text{ (i.e. } h(\| X \|) \text{ is the constant value of } f(X) \text{ for } X \in S_r, \text{)}
\]

If \( X \neq 0, \) show that \( h'(\| X \|) = g(X) \| X \|. \)

(c) Let \( D_r = \{ X \in \mathbb{R}^3 \mid \| X \| \leq r \} \) be the ball of radius \( r. \) Show that

\[
\int \int \int_{D_r} \text{div}(\nabla f) \, dV = 4\pi r^2 h'(r). \tag{4+3+3}\]
Solution:

(a) Let \( a, b \in S_r \) and \( R(t) \) be a differentiable curve in \( S_r \) joining \( a \) and \( b \). By the chain rule, 
\[
\frac{d}{dt} f(R(t)) = \nabla f(R(t)).R'(t) = g(R(t)).R'(t).
\]
Since \( R(t).R(t) = r^2 \), we get 
\[
R(t).R'(t) = 0 \implies \frac{d}{dt} f(R(t)) = 0.
\]
Hence, \( f \) is a constant on \( S_r \).

(b) \( f(X) = h(\|X\|) = h(\sqrt{x^2 + y^2 + z^2}) \).
By the Chain rule \( \nabla f(X) = (f_x, f_y, f_z) = \frac{h'(\|X\|)}{\|X\|}X \).
Hence \( \nabla f(X) = g(X)X \implies g(X) = h'(\|X\|) \).

(c) By the divergence theorem
\[
\int \int \int_D \text{div}(\nabla f) \, dV = \int \int_{S_r} \nabla f. n \, d\sigma
\]
\[
= \int \int_{S_r} \nabla f. \frac{X}{\|X\|} \, d\sigma = \int \int_{S_r} g(X) \|X\| \, d\sigma
\]
\[
= h'(r) \int_{S_r} d\sigma = 4\pi r^2 h'(r).
\]