MTH101A (2016), Tentative Marking Scheme - End sem. exam

1. (a) *Let $f(x,y,z) = xyz$ and S be $x^2+y^2+z^2 = 6$. Using Lagrange multipliers method, find the maximum and minimum values of f on S.* *[7]*

Solution:

Lag. Eqns.: $yz = 2\lambda x, xz = 2\lambda y, xy = 2\lambda z, x^2+y^2+z^2 = 6$ [1]

Case I: $\lambda = 0 \Rightarrow$ the candidates $(\pm\sqrt{6},0,0), (0,\pm\sqrt{6},0), (0,0,\pm\sqrt{6})$ [2]

Case II: $\lambda \neq 0 \Rightarrow 3xyz = 12\lambda$ [1]

Hence $2\lambda x^2 = 4\lambda$ which implies $x = \pm\sqrt{2}$ [1]

Similarly $y = \pm\sqrt{2}$ and $z = \pm\sqrt{2}$.

The candidates are $(\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2})$. [1]

The max value is $2\sqrt{2}$ and min value is $-2\sqrt{2}$ [1]

(This one mark is not to be released if Case I is NOT considered)

(b) *Evaluate the volume of the solid which is common to the cylinders $x^2+y^2 = 4$ and $y^2+z^2 = 4$ using the method of double integrals.* *[5]*

Solution:

$$V = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2\sqrt{4-y^2}\,dxdy \quad [3]$$

$$= \int_{-2}^{2} 4(4-y^2)dy \quad [1]$$

$$= \frac{128}{3} \quad [1]$$

(c) *Let $f : [\frac{1}{2}, \frac{3}{2}] \to \mathbb{R}$ be differentiable. Show that there exist $c_1, c_2 \in [\frac{1}{2}, \frac{3}{2}]$ such that $f'(c_2) = c_2 f'(c_1)$.* *[5]*

Solution:

By CMVT $\frac{f(\frac{3}{2})-f(\frac{1}{2})}{\frac{3}{2}-\frac{1}{2}} = \frac{f'(c_1)}{1}, c_1 \in [\frac{1}{2}, \frac{3}{2}]$ [2]

and $\frac{f(\frac{3}{2})-f(\frac{1}{2})}{(\frac{3}{2})^2-(\frac{1}{2})^2} = \frac{f'(c_2)}{2c_2}, c_2 \in [\frac{1}{2}, \frac{3}{2}]$ [2]

Therefore $\frac{f'(c_1)}{2} = \frac{f'(c_2)}{2c_2}$ [1]

2. (a) *Let the surface S be part of $2x+3y-z = 0$ which lies inside the region bounded by $x = 1, x = 2, y = 0$ and $y = x$. Evaluate $\iint_S \frac{d\sigma}{\sqrt{x^2+y^2}}$.* *[6]*

Solution:

Let R be the projection of the given region and $f(x,y) = 2x+3y$.

Then $\iint\limits_S \frac{d\sigma}{\sqrt{x^2+y^2}} = \iint\limits_R \frac{\sqrt{1+f_x^2+f_y^2}\,dxdy}{\sqrt{x^2+y^2}} = \iint\limits_R \frac{\sqrt{14}dxdy}{\sqrt{x^2+y^2}}$ [2]

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\sec\theta} \frac{\sqrt{14}}{r} r dr d\theta \quad \text{(See Figure 2(a))} \quad [3]$$

$$= \sqrt{14}\log(1+\sqrt{2}) \quad [1]$$

(b) *Let $X_0, U \in \mathbb{R}^2$ where $\|U\| = 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at X_0. Prove that the directional derivative $D_{X_0} f(U)$ of f at X_0 in the direction U exists and $D_{X_0} f(U) = \nabla f(x_0) \cdot U$.* [5]

Solution:

Since f is differentiable at X_0, $\frac{f(X_0+H)-f(X_0)-\nabla f(X_0)\cdot H}{\|H\|} \to 0$ as $H \to 0$ [1]

For $H = tU, t \in \mathbb{R}, \|U\| = 1$ [1]

and $t \to 0$, $\frac{f(X_0+tU)-f(X_0)-\nabla f(X_0)\cdot tU}{|t|} \to 0.$ [1]

As $t \to 0$, $\frac{f(X_0+tU)-f(X_0)-\nabla f(X_0)\cdot tU}{t} \to 0.$ [1]

As $t \to 0$, $\frac{f(X_0+tU)-f(X_0)}{t} \to \nabla f(X_0) \cdot U.$ [1]

(c) *Let $a_n \geq 0$ and $\sum_{n=1}^{\infty}(n^3 a_n^2 - 1)$ converge. Verify whether $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ converges.* [6]

Solution:

Observe that $n^3 a_n^2 \to 1$ [2]

LCT with $\sum \frac{1}{n^2}$: [1]

$\frac{a_n/\sqrt{n}}{1/n^2} = n^{\frac{3}{2}} a_n \to 1$ [2]

The series converges. [1]

3. (a) *Let the curve C be described by $R(t) = ((\sin 3t)\cos t, (\sin 3t)\sin t), 0 \leq t \leq \frac{\pi}{3}$. Sketch C. Evaluate $\oint_C ydx + xdy$ and $\oint_C -ydx + xdy$ where C is oriented counterclockwise.* [8]

Solution:

Note that C is $r = \sin 3\theta,\ 0 \leq \theta \leq \frac{\pi}{3}$. [1]

For the curve (see Figure 3(a)). [1]

Observe that $\oint_C ydx + xdy = \oint_C \nabla(xy) \cdot dR = 0$ by FTC of line integrals [2]

Observe that $\oint_C -ydx + xdy = 2(\text{Area enclosed by C})$. [2]

Area $= \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin^2(3\theta) d\theta$ [1]

$= \frac{\pi}{12}$. [1]

(b) *Let D be the region that lies below the surface $x^2+y^2+z^2 = 4z$ and above $z = \sqrt{3(x^2+y^2)}$. Using the spherical coordinates express $\iiint_D \sqrt{x^2+y^2+z^2}dV$ as three iterated single integrals.* [4]

Solution:

The sphere is $\rho^2 = 4\rho\cos\phi$, i.e., $\rho = 4\cos\phi$. [1]

The cone is $\rho\cos\phi = \sqrt{3}r = \sqrt{3}\rho\sin\phi$, i.e., $\phi = \frac{\pi}{6}$. [1]

Therefore $\iiint_D \sqrt{x^2+y^2+z^2}dV = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^{4\cos\phi} \rho\rho^2 \sin\phi d\rho d\phi d\theta$. [2]

(c) *Let* (a_n) *be in* $(0,1)$ *and* $4a_n(1-a_{n+1}) > 1$ *for all* $n \geq 1$. *Discuss the convergence/divergence of the series* $\sum_{n=1}^{\infty}(a_n^2 - 1)$. *[5]*

Solution:

If $a_n \to a_0$ for some a_0, then $4a_0(1-a_0) \geq 1$. [2]

Since $(2a_0-1)^2 \leq 0$, $a_0 = \frac{1}{2}$. [2]

Since $a_n^2 \nrightarrow 1$, $\sum(a_n^2 - 1)$ does NOT converge. [1]

4. (a) *Consider the arc* $(x-2)^2 + y^2 = 4, y \geq 0$. *Using a theorem of Pappus, find the surface area of the surface generated by revolving the arc about the line* $y + 2x = 0$. *[6]*

Solution:

Let the coordinate of the centroid of the arc be $(2, y_0)$. [1]

By Pappus theorem, $4\pi 2^2 = 2\pi y_0 2\pi$ [2]

Hence $y_0 = \frac{4}{\pi}$. [1]

Distance of the line from the centroid is $\rho = \frac{2\times 2 + y_0}{\sqrt{1+2^2}}$ [1]

By Pappus theorem, the required area is $2\pi\rho\pi 2$. [1]

(b) *Find the equation of the surface generated by the normals to the surface* $y + 2xz + xyz^2 = 0$ *at all points on the z-axis.* *[5]*

Solution:

Normal is $(2z + yz^2, 1 + xz^2, 2x + 2xyz)$. [1]

Normal at $(0,0,z_0)$ is $(2z_0, 1, 0)$. [1]

If (x,y,z) lies on the surface then, $\frac{x}{2z_0} = \frac{y}{1}, z = z_0$. [2]

The equation of the surfaces is $x = 2zy$. [1]

(c) *Let* $f : [0,\infty) \to [0,\infty)$ *be such that* $f''(x) > 0$ *for every* $x \geq 0$ *and* $\int_0^\infty f(x)dx$ *converges. Show that* $\int_0^n f(x)dx \geq nf(\frac{n}{2})$ *and* $f(\frac{n}{2}) \to 0$. *[6]*

Solution:

By Taylor's theorem, for $x \in [0,n]$, $f(x) \geq f(\frac{n}{2}) + f'(\frac{n}{2})(x - \frac{n}{2})$. [2]

Hence $\int\limits_0^n f(x)dx \geq nf(\frac{n}{2}) + f'(\frac{n}{2})\frac{n^2}{2} - f'(\frac{n}{2})\frac{n^2}{2}$. [2]

Since $\int\limits_0^\infty f(x)dx$ converges, there exists $M > 0$ such that $f(\frac{n}{2}) \leq \frac{M}{n}\ \forall\ n$. [2]

5. (a) *For* $p > 1$, *consider the curve* $C : |x|^p + |y|^p = 1$. *Evaluate* $\oint_C (\frac{-y}{x^2+y^2} + e^x(sinx))dx + (\frac{x}{x^2+y^2} + y(siny))dy$ *where* C *is oriented counter-clockwise.* *[6]*

Solution:

Given integral is $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy + \oint_C (e^x(sinx))dx + (y(siny))dy$. [1]

By Green's theorem, $\oint_C (e^x(sinx))dx + (y(siny))dy = 0.$ [1]

Observe that $\frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) = \frac{\partial}{\partial y}(\frac{-y}{x^2+y^2}).$ [1]

By Green's theorem, $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \oint_{C_r} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$ [2]

where C_r is a circle of radius r and C_r lies inside the region enclosed by C.

Hence the required value is $\int_0^{2\pi} \frac{-r\sin t d(r\cos t) + r\cos t d(r\sin t)}{r^2} = 2\pi.$ [1]

(b) *Consider the surface $S : x^2 + y^2 + z^2 = 8, -1 \leq z \leq 2.$* *[10]*

i. *Find a vector field F such that $curlF = (0, 0, 2\sqrt{8})$.*

ii. *Find the unit (outward) normal to S.*

iii. *If C_1 is $x^2 + y^2 = 4, z = 2$ then evaluate $|\oint_{C_1} F \cdot dR|$.*

iv. *Evaluate $\iint_S 2z d\sigma$.*

Solution:

(i) $F(x, y, z) = (-y\sqrt{8}, x\sqrt{8}, 0).$ [1]

(ii) The normal $\hat{n} = \frac{1}{\sqrt{8}}(x, y, z).$ [1]

(iii) Parametrization of C_1 is $(2\cos\theta, 2\sin\theta, 2),\ 0 \leq \theta \leq 2\pi$ [1]

The value of the line integral is $\int_0^{2\pi} -2\sqrt{8}\sin\theta d(2\cos\theta) + 2\sqrt{8}\cos\theta d(2\sin\theta)$

$= 8\sqrt{8}\pi$ [2]

(iv) Observe that $\iint_S 2z d\sigma = \iint_S curlF \cdot \hat{n} d\sigma.$ [1]

By Stoke's theorem, $\iint_S curlF \cdot \hat{n} d\sigma = (\oint_C - \oint_{C_1})F \cdot dR.$ [2]

where $C : x^2 + y^2 = 7, z = -1.$

$\oint_C F \cdot dR = 14\sqrt{8}\pi.$ [1]

Hence $\iint_S 2z d\sigma = 6\sqrt{8}\pi$ [1]

6. (a) *Sketch the graph of $f(x) = \frac{3x^2-2}{x^2-1}$ after finding the intervals of decreasing/increasing, intervals of concavity/convexity, points of local maximum and asymptotes.* *[5]*

Solution:

$f(x) = 3 + \frac{1}{x^2-1} \Rightarrow x = 1, x = -1$ and $y = 3$ are the asymptotes. [1]

$f'(x) = \frac{-2x}{(x^2-1)^2} \Rightarrow f$ is $\uparrow$ on $(-\infty, -1)$, $(-1, 0)$ and $\downarrow$ on $(0, 1), (1, \infty)$. [1]

$f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3} \Rightarrow$ convex on $(-\infty, -1), (1, \infty)$ and concave on $(-1, 1)$ [1]

$x = 0$ is a point of local maximum. [1]

For the graph (see Figure 6(a)). [1]

(b) *Consider the surfaces* *[11]*

$S_1 = \{(x, y, x + 100) : x^2 + y^2 \leq \frac{1}{9}\}$ *and*

$S_2 = \{(x, y, -100) : x^2 + y^2 \leq \frac{1}{9}\}$.

Let the surface S_3 *be the part of the cylinder* $x^2 + y^2 = \frac{1}{9}$ *that lies between the surfaces* S_1 *and* S_2. *Let* D *denote the region enclosed by* S_1, S_2 *and* S_3. *Let* $F(x, y, z) = \rho^{-3}(x, y, z)$ *for* $(x, y, z) \neq 0$ *where* $\rho = \sqrt{x^2 + y^2 + z^2}$.

i. *Find the unit normals to the surfaces* S_1, S_2 *and* S_3.

ii. *Find* $DivF$.

iii. *Evaluate*

$$\iint_{S_1} \frac{(z-x)d\sigma}{(\sqrt{2})\rho^3} + \iint_{S_2} \frac{-zd\sigma}{\rho^3} + \iint_{S_3} \frac{3(x^2+y^2)d\sigma}{\rho^3}.$$

Solution:

(i) Unit normal on $S_1 : \widehat{n_1} = \frac{1}{\sqrt{2}}(-1, 0, 1)$ or $(-\widehat{n_1})$. [1]

Unit normal on $S_2 : \widehat{n_2} = (0, 0, -1)$ or $(-\widehat{n_2})$. [1]

Unit normal on $S_3 : \widehat{n_3} = (3x, 3y, 0)$ or $(-\widehat{n_3})$. [1]

(ii) $\frac{\partial}{\partial x}(\rho) = \frac{x}{\rho}$

$\frac{\partial}{\partial x}(\frac{x}{\rho^3}) = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}$ [2]

$divF = \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0$

(iii) The given integral $I = \iint_{S_1} F \cdot \widehat{n_1} d\sigma + \iint_{S_2} F \cdot \widehat{n_2} d\sigma + \iint_{S_3} F \cdot \widehat{n_3} d\sigma$. [2]

By divergence theorem $I = \iint_{S} F \cdot \widehat{n} d\sigma$ [1]

where $S : x^2 + y^2 + z^2 = r^2, r < \frac{1}{3}$ and $\widehat{n} = \frac{1}{r}(x, y, z)$. [1]

$I = 4\pi$. [2]