1. (a) Let \( f(x, y, z) = xyz \) and \( S \) be \( x^2 + y^2 + z^2 = 6 \). Using Lagrange multipliers method, find the maximum and minimum values of \( f \) on \( S \). [7]

Solution:
Lag. Eqns.: \( yz = 2\lambda x, xz = 2\lambda y, xy = 2\lambda z, x^2 + y^2 + z^2 = 6 \) \[1\]
Case I: \( \lambda = 0 \) ⇒ the candidates \((±\sqrt{6}, 0, 0), (0, ±\sqrt{6}, 0), (0, 0, ±\sqrt{6})\) \[2\]
Case II: \( \lambda ≠ 0 \) ⇒ \( 3xyz = 12\lambda \)
Hence \( 2\lambda x^2 = 4\lambda \) which implies \( x = ±\sqrt{2} \) \[1\]
Similarly \( y = ±\sqrt{2} \) and \( z = ±\sqrt{2} \).
The candidates are \((±\sqrt{2}, ±\sqrt{2}, ±\sqrt{2})\). \[1\]
The max value is \( 2\sqrt{2} \) and min value is \(-2\sqrt{2} \)
(One mark is not to be released if Case I is NOT considered)

(b) Evaluate the volume of the solid which is common to the cylinders \( x^2 + y^2 = 4 \) and \( y^2 + z^2 = 4 \) using the method of double integrals. [5]

Solution:
\[
V = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2\sqrt{4-y^2} dy \, dx
\]
\[
= \int_{-2}^{2} 4(4-y^2) \, dy
\]
\[
= \frac{128}{3}
\]

(c) Let \( f : [\frac{1}{2}, \frac{3}{2}] \to \mathbb{R} \) be differentiable. Show that there exist \( c_1, c_2 \in [\frac{1}{2}, \frac{3}{2}] \) such that \( f'(c_2) = c_2 f'(c_1) \). [5]

Solution:
By CMVT \( \frac{f(\frac{3}{2}) - f(\frac{1}{2})}{\frac{3}{2} - \frac{1}{2}} = \frac{f'(c_1)}{1}, c_1 \in [\frac{1}{2}, \frac{3}{2}] \) \[2\]
and \( \frac{f(\frac{3}{2}) - f(\frac{1}{2})}{(\frac{3}{2})^2 - (\frac{1}{2})^2} = \frac{f'(c_2)}{1}, c_2 \in [\frac{1}{2}, \frac{3}{2}] \) \[2\]
Therefore \( \frac{f'(c_1)}{2} = \frac{f'(c_2)}{2x^2} \)

2. (a) Let the surface \( S \) be part of \( 2x + 3y - z = 0 \) which lies inside the region bounded by \( x = 1, x = 2, y = 0 \) and \( y = x \). Evaluate \( \iint_S \frac{dz}{\sqrt{x^2 + y^2}} \). [6]

Solution:
Let \( R \) be the projection of the given region and \( f(x, y) = 2x + 3y \).
Then \( \iint_R \frac{dz}{\sqrt{x^2 + y^2}} = \iint_R \frac{1}{\sqrt{1+f_x^2 + f_y^2}} \, dxdy \)
\[
= \int_{0}^{\pi/4} \int_{\sec \theta}^{\tan \theta} \frac{\sqrt{x^2+y^2}}{r} \, dr \, d\theta \quad \text{(See Figure 2(a))}
\]
\[
= \sqrt{14} \log(1 + \sqrt{2})
\]
(b) Let $X_0, U \in \mathbb{R}^2$ where $\|U\| = 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $X_0$. Prove that the directional derivative $D_{X_0}f(U)$ of $f$ at $X_0$ in the direction $U$ exists and $D_{X_0}f(U) = \nabla f(x_0) \cdot U$. [5]

**Solution:**
Since $f$ is differentiable at $X_0$, $f(X_0 + H) - f(X_0) - \nabla f(X_0) \cdot H \to 0$ as $H \to 0$ [1]

For $H = tU, t \in \mathbb{R}, \|U\| = 1$

and $t \to 0, \frac{f(X_0 + tU) - f(X_0) - \nabla f(X_0) \cdot tU}{t} \to 0$.

As $t \to 0, \frac{f(X_0 + tU) - f(X_0) - \nabla f(X_0) \cdot tU}{t} \to 0$.

As $t \to 0, \frac{f(X_0 + tU) - f(X_0)}{t} \to \nabla f(X_0) \cdot U$.

(c) Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} (n^3 a_n^2 - 1)$ converge. Verify whether $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ converges. [6]

**Solution:**
Observe that $n^3 a_n^2 \to 1$ [2]

LCT with $\sum \frac{1}{n^2}$:

$\frac{a_n/\sqrt{n}}{1/n^2} = n^{3/2} a_n \to 1$

The series converges. [1]

3. (a) Let the curve $C$ be described by $R(t) = ((\sin 3t) \cos t, (\sin 3t) \sin t), 0 \leq t \leq \frac{\pi}{3}$. Sketch $C$. Evaluate $\oint_C ydx + xdy$ and $\oint_C -ydx + xdy$ where $C$ is oriented counterclockwise. [8]

**Solution:**
Note that $C$ is $r = \sin 3\theta$, $0 \leq \theta \leq \frac{\pi}{3}$. [1]

For the curve (see Figure 3(a)). [1]

Observe that $\oint_C ydx + xdy = \oint_C \nabla(xy) \cdot dR = 0$ by FTC of line integrals [2]

Observe that $\oint_C -ydx + xdy = 2(\text{Area enclosed by } C)$. [2]

Area $= \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin^2(3\theta) d\theta$

$= \frac{\pi}{12}$. [1]

(b) Let $D$ be the region that lies below the surface $x^2 + y^2 + z^2 = 4z$ and above $z = \sqrt{3(x^2 + y^2)}$. Using the spherical coordinates express $\iiint_D \sqrt{x^2 + y^2 + z^2} dV$ as three iterated single integrals. [4]

**Solution:**
The sphere is $\rho^2 = 4\rho \cos \phi$, i.e., $\rho = 4 \cos \phi$. [1]

The cone is $\rho \cos \phi = \sqrt{3} r = \sqrt{3} \rho \sin \phi$, i.e., $\phi = \frac{\pi}{6}$. [1]

Therefore $\iiint_D \sqrt{x^2 + y^2 + z^2} dV = \int_0^{\frac{\pi}{3}} \int_0^{\frac{4\cos \phi}{\rho}} \int_0^{2\rho} \rho \rho^2 \sin \phi d\rho d\phi d\theta$. [2]
(c) Let \((a_n)\) be in \((0,1)\) and \(4a_n(1-a_{n+1}) > 1\) for all \(n \geq 1\). Discuss the convergence/divergence of the series \(\sum_{n=1}^{\infty} (a_n^2 - 1)\). [5]

Solution:
If \(a_n \to a_0\) for some \(a_0\), then \(4a_0(1-a_0) \geq 1\). [2]
Since \((2a_0 - 1)^2 \leq 0\), \(a_0 = \frac{1}{2}\). [2]
Since \(a_n^2 \to 1\), \(\sum(a_n^2 - 1)\) does NOT converge. [1]

4. (a) Consider the arc \((x-2)^2 + y^2 = 4, y \geq 0\). Using a theorem of Pappus, find the surface area of the surface generated by revolving the arc about the line \(y + 2x = 0\). [6]

Solution:
Let the coordinate of the centroid of the arc be \((2\bar{y}, z_0)\). [1]
By Pappus theorem, \(4\pi \bar{y} = 2\pi y_0^2\pi\). [2]
Hence \(\bar{y} = \frac{y_0}{4}\). [1]
The distance of the line from the centroid is \(\rho = \sqrt{\frac{2y_0^2 + y_0}{1+2\bar{y}}}\). [1]
By Pappus theorem, the required area is \(2\pi \rho \pi 2\). [1]

(b) Find the equation of the surface generated by the normals to the surface \(y + 2xz + xyz^2 = 0\) at all points on the \(z\)-axis. [5]

Solution:
Normal is \((2z + yz^2, 1 + xz^2, 2x + 2xyz)\). [1]
Normal at \((0, 0, z_0)\) is \((2z_0, 1, 0)\). [1]
If \((x, y, z)\) lies on the surface then, \(x^2 z_0 = y, z = z_0\). [2]
The equation of the surfaces is \(x = 2zy\). [1]

(c) Let \(f: [0, \infty) \to [0, \infty)\) be such that \(f''(x) > 0\) for every \(x \geq 0\) and \(\int_0^{\infty} f(x)dx\) converges. Show that \(\int_0^{n} f(x)dx \geq nf(\frac{n}{2})\) and \(f(\frac{n}{2}) \to 0\). [6]

Solution:
By Taylor’s theorem, for \(x \in [0, n]\), \(f(x) \geq f(\frac{n}{2}) + f'(\frac{n}{2})(x - \frac{n}{2})\). [2]
Hence \(\int_0^{n} f(x)dx \geq nf(\frac{n}{2}) + f'(\frac{n}{2}) \frac{n^2}{2} - f'(\frac{n}{2}) \frac{n^2}{2}\). [2]
Since \(\int_0^{\infty} f(x)dx\) converges, there exists \(M > 0\) such that \(f(\frac{n}{2}) \leq \frac{M}{n}\ \forall \ n\). [2]

5. (a) For \(p > 1\), consider the curve \(C: |x|^p + |y|^p = 1\). Evaluate \(\oint_{C} \left(\frac{y}{x+y} + e^x(sinx)\right)dx + \left(\frac{x}{x+y} + y(siny)\right)dy\) where \(C\) is oriented counterclockwise. [6]

Solution:
Given integral is \(\oint_{C} \frac{y}{x+y}dx + \frac{x}{x+y}dy + \oint_{C}(e^x(sinx))dx + (y(siny))dy\). [1]
By Green’s theorem, \[ \oint_C (e^x \sin x) \, dx + (y \sin y) \, dy = 0. \] [1]

Observe that \[ \frac{\partial}{\partial x} \left( \frac{x}{x+y^2} \right) = \frac{\partial}{\partial y} \left( \frac{-y}{x+y^2} \right). \] [1]

By Green’s theorem, \[ \oint_C \frac{-y}{x+y^2} \, dx + \frac{x}{x+y^2} \, dy = \oint_{C_r} \frac{-y}{x+y^2} \, dx + \frac{x}{x+y^2} \, dy, \] where \( C_r \) is a circle of radius \( r \) and \( C_r \) lies inside the region enclosed by \( C \).

Hence the required value is \[ \int_0^{2\pi} \frac{-r \sin t \cos t + r \cos t \sin t}{r^2} \, dt = 2\pi. \] [1]

(b) Consider the surface \( S : x^2 + y^2 + z^2 = 8, -1 \leq z \leq 2. \) [10]

i. Find a vector field \( F \) such that \( \text{curl} F = (0, 0, 2\sqrt{8}). \)

ii. Find the unit (outward) normal to \( S \).

iii. If \( C_1 \) is \( x^2 + y^2 = 4, z = 2 \) then evaluate \( \oint_{C_1} F \cdot dR \).

iv. Evaluate \( \iint_S 2z \, d\sigma \).

**Solution:**

(i) \( F(x, y, z) = (-y\sqrt{8}, x\sqrt{8}, 0). \) [1]

(ii) The normal \( \mathbf{n} = \frac{1}{\sqrt{8}}(x, y, z). \) [1]

(iii) Parametrization of \( C_1 \) is \( (2 \cos \theta, 2 \sin \theta, 2), \quad 0 \leq \theta \leq 2\pi \)

The value of the line integral is \[ \int_0^{2\pi} -2\sqrt{8} \sin \theta \, d(2 \cos \theta) + 2\sqrt{8} \cos \theta \, d(2 \sin \theta) \]

\[ = 8\sqrt{8}\pi \] [2]

(iv) Observe that \( \iint_S 2z \, d\sigma = \iint_S \text{curl} F \cdot \mathbf{n} \, d\sigma. \) [1]

By Stoke’s theorem, \( \iint_S \text{curl} F \cdot \mathbf{n} \, d\sigma = (\oint_C - F) \cdot dR. \) [2]

where \( C : x^2 + y^2 = 7, z = -1. \)

\[ \oint_C F \cdot dR = 14\sqrt{8}\pi. \] [1]

Hence \( \iint_S 2z \, d\sigma = 6\sqrt{8}\pi \) [1]

6. (a) Sketch the graph of \( f(x) = \frac{3x^2 - 2}{x^2 - 1} \) after finding the intervals of decreasing/increasing, intervals of concavity/convexity, points of local maximum and asymptotes. [5]

**Solution:**

\( f(x) = 3 + \frac{1}{x^2 - 1} \Rightarrow x = 1, x = -1 \) and \( y = 3 \) are the asymptotes. [1]

\( f'(x) = \frac{-2x}{(x^2-1)^2} \Rightarrow f \) is \( \uparrow \) on \( (-\infty, -1), (-1, 0) \) and \( \downarrow \) on \( (0, 1), (1, \infty) \). [1]

\( f''(x) = \frac{2(3x^2 + 1)}{(x^2 - 1)^3} \Rightarrow \) convex on \( (-\infty, -1), (1, \infty) \) and concave on \( (-1, 1) \) [1]

\( x = 0 \) is a point of local maximum. [1]

For the graph (see Figure 6(a)). [1]
(b) Consider the surfaces

\[ S_1 = \{(x, y, x + 100) : x^2 + y^2 \leq \frac{1}{9}\} \text{ and } \]
\[ S_2 = \{(x, y, -100) : x^2 + y^2 \leq \frac{1}{9}\}. \]

Let the surface \( S_3 \) be the part of the cylinder \( x^2 + y^2 = \frac{1}{9} \) that lies between the surfaces \( S_1 \) and \( S_2 \). Let \( D \) denote the region enclosed by \( S_1, S_2 \) and \( S_3 \). Let \( F(x, y, z) = \rho^{-3}(x, y, z) \) for \( (x, y, z) \neq 0 \) where \( \rho = \sqrt{x^2 + y^2 + z^2} \).

i. Find the unit normals to the surfaces \( S_1, S_2 \) and \( S_3 \).

ii. Find \( \text{Div} F \).

iii. Evaluate

\[ \iint_{S_1} \frac{(z-x)d\sigma}{\rho^3} + \iint_{S_2} \frac{z d\sigma}{\rho^3} + \iint_{S_3} 3(x^2+y^2) d\sigma \rho^3. \]

Solution:

(i) Unit normal on \( S_1 \) : \( \hat{n}_1 = \frac{1}{\sqrt{2}}(-1, 0, 1) \) or \((-\hat{n}_1)\). [1]

Unit normal on \( S_2 \) : \( \hat{n}_2 = (0, 0, -1) \) or \((-\hat{n}_2)\). [1]

Unit normal on \( S_3 \) : \( \hat{n}_3 = (3x, 3y, 0) \) or \((-\hat{n}_3)\). [1]

(ii) \( \frac{\partial}{\partial x}(\rho) = \frac{z}{\rho} \)

\( \frac{\partial}{\partial x}(\frac{z}{\rho}) = \frac{1}{\rho^2} - \frac{3x^2}{\rho^3} \) [2]

\( \text{Div} F = \frac{3}{\rho^3} - \frac{3y^2}{\rho^3} = 0 \)

(iii) The given integral \( I = \iint_{S_1} F \cdot \hat{n}_1 d\sigma + \iint_{S_2} F \cdot \hat{n}_2 d\sigma + \iint_{S_3} F \cdot \hat{n}_3 d\sigma. \) [2]

By divergence theorem \( I = \iint_{S} F \cdot \hat{n} d\sigma \) [1]

where \( S : x^2 + y^2 + z^2 = r^2, r < \frac{1}{3} \) and \( \hat{n} = \frac{1}{r}(x, y, z). \) [1]

\( I = 4\pi. \) [2]