(1) (a) Let \( x_n = \left( \frac{a^n + b^n + c^n}{3} \right)^{\frac{1}{n}} + \frac{n^a}{(1+b+c)^n} \) where 1 < a < b < c. Show that the sequence \((x_n)\) converges and find its limit. \[5\]

**Solution:** Note that \( \left( \frac{c^n}{3} \right)^{\frac{1}{n}} \leq \left( \frac{a^n + b^n + c^n}{3} \right)^{\frac{1}{n}} \leq \left( \frac{3c^n}{3} \right)^{\frac{1}{n}} \). \[1\]
By Sandwich Theorem, \( \left( \frac{a^n + b^n + c^n}{3} \right)^{\frac{1}{n}} \to c \). \[1\]

Let \( y_n = \frac{n^a}{(1+b+c)^n} \). Then \( y_{n+1}/y_n \to \frac{1}{1+b+c} < 1 \). \[1\]
Hence \( y_n \to 0 \). \[1\]
Therefore \( x_n \to c \). \[1\]

(b) Let \((a_n)\) be a sequence defined by
\[ a_1 = 2, \quad a_2 = 4 \quad \text{and} \quad a_{n+2} = \frac{1}{4}a_n + \frac{3}{4}a_{n+1} \quad \text{for} \quad n \geq 1. \]

Does the series \( \sum_{n=1}^{\infty} a_n \) converge? Justify your answer. \[4\]

**Solution:** Observe that \( 2 \leq a_n \) for all \( n \).
Hence \( a_n \to 0 \). \[2\]
Therefore the series does not converge. \[1\]

(c) Let \( f : [0, 2] \to \mathbb{R} \) be a differentiable function such that \( f(1 + \frac{1}{n}) = 0 \) for all \( n \in \mathbb{N} \). Show that \( f'(1) = 0 \). \[3\]

**Solution:** Since \( f(1 + \frac{1}{n}) \to f(1), \) \( f(1) = 0 \).
Note that \( f'(1) = \lim_{n \to \infty} \frac{f(1 + \frac{1}{n}) - f(1)}{1/n} = 0. \) \[2\]

(2) (a) Determine the values of \( x \) for which the series \( \sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^3} \) converges. \[4\]

**Solution:** By the root test, the series converges for \( |x - 1| < \sqrt{3} \).
The series converges for \( |x - 1| = \sqrt{3} \). \[2\]
The series converges for \( |x - 1| \leq \sqrt{3} \).

(b) Test the convergence/divergence of the improper integral \( \int_1^{\infty} (-t^3)e^{-2t} \, dt \). \[4\]

**Solution:** Note that \( \lim_{t \to \infty} \frac{t^3e^{-2t}}{1/t^2} = 0. \) \[2\]
By the LCT \( \int_1^{\infty} t^3e^{-2t} \, dt \) converges.
Hence the given improper integral converges. \[1\]
(c) Sketch the graphs of the following polar equations:

(i) \( r^2 = -5\cos\theta \)  
(ii) \( r^2 = -2\sin 2\theta \).

(3) (a) The region bounded by the curves \( y = \sqrt{1 - x^2}, y = \sqrt{4 - x^2} \) and the x-axis is revolved around the axis \( y = -1 \). Find the volume generated.

**Solution:** Let \((0, \bar{y})\) be the centroid of the given region.
Then by Pappus theorem \( \frac{4}{3}\pi(2^3 - 1) = 2\pi\bar{y}\pi\frac{4-1}{2} \).
Hence \( \bar{y} = \frac{3\pi}{8} \).
The required volume, by Pappus theorem, is \( V = 2\pi(\bar{y} + 1)\pi\frac{3}{2}\).

(b) Find the curvature \( \kappa(t) \) of the curve defined by the parametric equations:

\[
\begin{align*}
x(t) &= \int_0^t \cos \frac{u^2}{2} \, du, \\
y(t) &= \int_0^t \sin \frac{u^2}{2} \, du, \\t &> 0.
\end{align*}
\]

**Solution:**
\[
\begin{align*}
v(t) &= (\cos \frac{t^2}{2}, \sin \frac{t^2}{2}) \\
a(t) &= (-t\sin \frac{t^2}{2}, t\cos \frac{t^2}{2}) \\
\kappa(t) &= \frac{\|a(t) \times v(t)\|}{\|v(t)\|^3} = t.
\end{align*}
\]

(c) Let \( f(x, y) = x^2 + y^2 + x + y \). What is the directional derivative of \( f \) at \( (0, 0) \) in the direction \((\frac{3}{5}, \frac{4}{5})\)?

**Solution:**
\[
D_{(0,0)}f(\frac{3}{5}, \frac{4}{5}) = \nabla f(0, 0) \cdot (\frac{3}{5}, \frac{4}{5}) = (\frac{3}{5}, \frac{4}{5}).
\]

(4) (a) Find the area of the region bounded between the circles \( x^2 + y^2 = 2x \) and \( x^2 + y^2 = 4x \) using double integral.

**Solution:**
Let \( R \) denote the region bounded by the circles.
Then the required area \( A = \int \int_R dA = \int_{-\pi/2}^{\pi/2} \int_{2\cos\theta}^{4\cos\theta} r\, dr\, d\theta \)
\[
= 6 \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta \\
= 6 \left[ \frac{1}{4}\sin 2\theta + \frac{x}{2}\right]_{-\pi/2}^{\pi/2} = 3\pi
\]

**No marks will be awarded if double integral is not used.**

(b) Determine the equation of the cone with vertex \((0, -a, 0)\) generated by a line passing through the curve \( z^2 = 2y, x = h \) where \( a \) and \( h \) are positive constants.

**Solution:**
Any point on the curve is of the form \((h, y_0, z_0)\).
The equation of a line passing through \((h, y_0, z_0)\) and \((0, -a, 0)\) is \( \frac{x-h}{h-0} = \frac{y+y_0}{y_0+a} = \frac{z-a}{z_0} \).
We get \( z_0 = \frac{hx}{x} \) and \( y_0 = \frac{h(y+z)}{x} - a \).
Since \((h, y_0, z_0)\) lies on the curve, the equation of the cone is \( h^2z^2 = 2x[h(y + a) - ax] \).

(c) Let \( f(x, y) = \sin(x) - \cos(y) \). Find the derivative of \( f \) at \((0, 0)\).
(5) (a) Let \( f(x, y) = y^4 - 4y^2x + x^2 \). Show that \((0, 0)\) is a saddle point.

**Solution:** The derivative of \( f \) at \((0, 0)\) is \((f_x, f_y)(0, 0) = (1, 0)\).

(b) Let \( f(x, y) = x^5 - 3xy^2 \). Find the points of local maxima, local minima and saddle points.

**Solution:** \( f_x = f_y = 0 \) implies that \((0, 0)\) is the only critical point. 
Note that \((f_{xx} f_{yy} - f_{xy}^2)(0, 0) = 0\) and hence the second derivative fails. 
Hence for \( x > 0 \), \( f(x, 0) > 0 \) and \( f(-x, 0) < 0 \) and therefore \((0, 0)\) is a saddle point.

(c) Let \( C \) be the line segment joining \((0, 0, 0)\) and \((1, 1, 1)\). Let \( f(x, y, z) = (x^2, y^2, z^2) \) for all \((x, y, z) \in C\). Compute the line integral \( \int_C f \cdot dR \) for all \((x, y, z) \in C\). 

**Solution:** \( f_x = f_y = 0 \) implies that \((0, 0)\) is the only critical point.
Note that \((f_{xx} f_{yy} - f_{xy}^2)(0, 0) = 0\) and hence the second derivative fails. 
Hence for \( x > 0 \), \( f(x, 0) > 0 \) and \( f(-x, 0) < 0 \) and therefore \((0, 0)\) is a saddle point.

(6) (a) Evaluate \( \int_0^1 \int_0^{\sqrt{x^2} + y^2} e^{y^3} dy dx dz \).

**Solution:** Note that \( \int_0^1 \int_0^{\sqrt{x^2} + y^2} e^{y^3} dy dx = \int_0^1 e^{y^3} dy = 1 \frac{1}{3}(e - 1) \). 
Hence \( \int_0^1 \int_0^{\sqrt{x^2} + y^2} e^{y^3} dy dx dz = \frac{1}{3}(e - 1) \).

(b) A round hole of radius \( \sqrt{3} \) cms is bored through the center of a solid sphere of radius 2 cms. Evaluate the volume cut out using double integral. Leave your answer with the integral expressions (i.e., there is no need to evaluate the integrals).

**Solution:** \( V = \int_{\sqrt{3}}^{\sqrt{3 + x^2}} \int_{\sqrt{3 - x^2}}^{\sqrt{3}} 2\sqrt{4 - x^2 - y^2} dy dx \) where \( R = \{(x, y) : x^2 + y^2 \leq 3\} \).

\[
= \int_{\sqrt{3}}^{\sqrt{3 + x^2}} \int_{\sqrt{3 - x^2}}^{\sqrt{3}} 2\sqrt{4 - x^2 - y^2} dy dx
\]

OR \( = \int_0^{\sqrt{3}} \int_0^{\sqrt{3 - r^2}} 2\sqrt{4 - r^2} dr d\theta \).

(c) Consider the surface \( z = \sqrt{1 - x^2 - y^2} \) where \( x^2 + y^2 \leq 1 \). Find the unit outward normal to the surface at the point \( \left( \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \right) \).

**Solution:** The unit outward normal is the same, i.e., \( \left( \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \right) \).
(7) (a) Let $f(x, y) = (xy^2 + \sin^2 x)i + (x^2y + 2x + \cos^2 y)j$. Find the line integral $\int_C f \cdot dR$ where $C$ is the triangle in the plane with the vertices $(0, 0), (0, 4)$ and $(1, 1)$ and $C$ is oriented counterclockwise. [4]

Solution: Let $K$ denote the region bounded by the given triangle.

By Green’s theorem $\int_C f \cdot dR = \int \int_K 2 \ dA$. [2]

$= 2 \ (\text{Area of the triangle }) = 4$. [2]

(b) Let $C_1$ be the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Let $C_2$ be the curve of intersection of the plane $y + z = 4$ and the cylinder $x^2 + y^2 = 4$. Suppose $C_1$ and $C_2$ are oriented counterclockwise when viewed from above. Let $F$ be a vector field with $\text{curl} F = (1, 0, 1)$. Show that $\int_{C_1} F \cdot dR = 4 \int_{C_2} F \cdot dR$. [6]

Solution: If the surface $S$ is defined by $z = f(x, y)$ then by the Stokes theorem $\int_C F \cdot dR = \int \int_S (\text{curl} F) \cdot \hat{n} d\sigma = \int \int_K (\text{curl} F) \cdot \frac{-f_y^2 - f_z^2 + k}{\sqrt{f_x^2 + f_y^2 + 1}} d^2 + f^2 + 1 \ dxdy$. [2]

Here the surfaces are portions from $z = 2 - y$ and $z = 4 - y$.

Hence $\int_{C_1} F \cdot dR = \int \int_{x^2 + y^2 \leq 1} (1, 0, 1) \cdot (0, 1, 1) dxdy = \pi$. [2]

$\int_{C_2} F \cdot dR = \int \int_{x^2 + y^2 \leq 4} (1, 0, 1) \cdot (0, 1, 1) dxdy = 4\pi$. [2]