END SEMESTER EXAMINATION

MTH 101A, 19TH DECEMBER 2020

Tentative Marking Scheme

Time: 3 hrs (16:00-19:00hrs) **Full Marks: 80**

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(1) (a) Let $x_n = \left(\frac{a^n+b^n+c^n}{3}\right)^{\frac{1}{n}} + \frac{n^a}{(1+b+c)^n}$ where $1 < a < b < c$. Show that the sequence (x_n) converges and find its limit. [5]

Solution: Note that $\left(\frac{c^n}{3}\right)^{\frac{1}{n}} \leq \left(\frac{a^n+b^n+c^n}{3}\right)^{\frac{1}{n}} \leq \left(\frac{3c^n}{3}\right)^{\frac{1}{n}}$. [1]
By Sandwich Theorem, $\left(\frac{a^n+b^n+c^n}{3}\right)^{\frac{1}{n}} \to c$. [1]
Let $y_n = \frac{n^a}{(1+b+c)^n}$. Then $\frac{y_{n+1}}{y_n} \to \frac{1}{1+b+c} < 1$. [1]
Hence $y_n \to 0$. [1]
Therefore $x_n \to c$. [1]

(b) Let (a_n) be a sequence defined by

$$a_1 = 2,\ a_2 = 4 \ \text{ and } \ a_{n+2} = \frac{1}{4}a_n + \frac{3}{4}a_{n+1} \ \text{ for } \ n \geq 1.$$

Does the series $\sum_{n=1}^{\infty} a_n$ converge? Justify your answer. [4]

Solution: Observe that $2 \leq a_n$ for all n. [2]
Hence $a_n \nrightarrow 0$. [1]
Therefore the series does not converge. [1]

(c) Let $f : [0,2] \to \mathbb{R}$ be a differentiable function such that $f(1+\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Show that $f'(1) = 0$. [3]

Solution: Since $f(1+\frac{1}{n}) \to f(1)$, $f(1) = 0$. [1]
Note that $f'(1) = \lim\limits_{n\to\infty} \dfrac{f(1+\frac{1}{n}) - f(1)}{1/n} = 0$. [2]

(2) (a) Determine the values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^4 3^n}$ converges. [4]

Solution: By the root test, the series converges for $|x-1| < \sqrt{3}$. [2]
The series converges for $|x-1| = \sqrt{3}$. [2]
The series converges for $|x-1| \leq \sqrt{3}$.

(b) Test the convergence/divergence of the improper integral $\int_1^{\infty} (-t^3)e^{-2t}dt$. [4]

Solution: Note that $\lim\limits_{t\to\infty} \dfrac{t^3 e^{-2t}}{1/t^2} = 0$. [2]
By the LCT $\int_1^{\infty} t^3 e^{-2t} dt$ converges. [1]
Hence the given improper integral converges. [1]

(c) Sketch the graphs of the following polar equations:
(i) $r^2 = -5cos\theta$ (ii) $r^2 = -2sin2\theta$. [4]

(3) (a) The region bounded by the curves $y = \sqrt{1-x^2}, y = \sqrt{4-x^2}$ and the x-axis is revolved around the axis $y = -1$. Find the volume generated. [5]

Solution: Let $(0, \overline{y})$ be the centroid of the given region.
Then by Pappus theorem $\frac{4}{3}\pi(2^3 - 1) = 2\pi\overline{y}\pi\frac{4-1}{2}$. [2]
Hence $\overline{y} = \frac{28}{9\pi}$. [1]
The required volume, by Pappus theorem, is $V = 2\pi(\overline{y} + 1)\pi\frac{3}{2}$. [2]

(b) Find the curvature $\kappa(t)$ of the curve defined by the parametric equations:

$$x(t) = \int_0^t cos\frac{u^2}{2}du, \quad y(t) = \int_0^t sin\frac{u^2}{2}du, \quad t > 0.$$ [5]

Solution: $v(t) = (cos\frac{t^2}{2}, sin\frac{t^2}{2})$ [2]
$a(t) = (-tsin\frac{t^2}{2}, tcos\frac{t^2}{2})$ [1]
$\kappa(t) = \frac{\|a(t)\times v(t)\|}{\|v(t)\|^3} = t$ [2]

(c) Let $f(x, y) = x^2 + y^2 + x + y$. What is the directional derivative of f at $(0, 0)$ in the direction $(\frac{3}{5}, \frac{4}{5})$?. [2]

Solution: $D_{(0,0)}f(\frac{3}{5}, \frac{4}{5}) = \nabla f(0, 0) \cdot (\frac{3}{5}, \frac{4}{5}) = (\frac{3}{5}, \frac{4}{5})$. [2]

(4) (a) Find the area of the region bounded between the circles $x^2 + y^2 = 2x$ and $x^2 + y^2 = 4x$ using double integral. [5]

Solution: Let R denote the region bounded by the circles.
Then the required area $A = \int\int_R dA = \int_{-\pi/2}^{\pi/2}\int_{2cos\theta}^{4cos\theta} rdrd\theta$ [2]
$= 6\int_{-\pi/2}^{\pi/2} cos^2 d\theta$ [2]
$= 6\left[\frac{1}{4}sin2x + \frac{x}{2}\right]_{-\pi/2}^{\pi/2} = 3\pi$ [1]

No marks will be awarded if double integral is not used.

(b) Determine the equation of the cone with vertex $(0, -a, 0)$ generated by a line passing through the curve $z^2 = 2y, x = h$ where a and h are positive constants. [5]

Solution: Any point on the curve is of the form (h, y_0, z_0).
The equation of a line passing through (h, y_0, z_0) and $(0, -a, 0)$ is
$\frac{x-0}{h-0} = \frac{y+a}{y_0+a} = \frac{z-0}{z_0}$. [2]
We get $z_0 = \frac{hz}{x}$ and $y_0 = \frac{h(y+a)}{x} - a$. [1]
Since (h, y_0, z_0) lies on the curve, the equation of the cone is
$h^2z^2 = 2x[h(y + a) - ax]$. [2]

(c) Let $f(x, y) = sin(x) - cos(y)$. Find the derivative of f at $(0, 0)$. [2]

Solution: The derivative of f at $(0,0) = (f_x, f_y)(0,0) = (1,0)$ [2]

(5) (a) Let $f(x,y) = y^4 - 4y^2x + x^2$. Show that $(0,0)$ is a saddle point. [4]

Solution: Note that $f_x(0,0) = 0$ and $f_y(0,0) = 0$ [1]
For any $\epsilon > 0$, $f(0,\epsilon) > 0$ [1]
and $f(2\epsilon^2, \epsilon) < 0$. [2]

(b) Let $f(x,y) = x^5 - 3xy^2$. Find the points of local maxima, local minima and saddle points. [5]

Solution: $f_x = f_y = 0$ implies that $(0,0)$ is the only critical point. [1]
Note that $(f_{xx}f_{yy} - f_{xy}^2)(0,0) = 0$ and hence the second derivative fails. [1]
Note that $f(x,0) = x^5$. [2]
Hence for $x > 0$, $f(x,0) > 0$ and $f(-x,0) < 0$ and therefore $(0,0)$ is a saddle point. [1]

(c) Let C be the line segment joining $(0,0,0)$ and $(1,1,1)$. Let $f(x,y,z) = (x^2, y^2, z^2)$ for all $(x,y,z) \in C$. Compute the line integral $\oint_C f \cdot dR$ [3]

Solution: Note that $C = R(t) = (t,t,t), t \in [0,1]$. [1]

Therefore $\int_C f \cdot dR = \int_0^1 t^2dt + t^2dt + t^2dt$ [2]

(6) (a) Evaluate $\int_0^1 \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydxdz$. [4]

Solution: Note that $\int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydx = \int_0^1 \int_0^{y^2} e^{y^3} dxdy$ [2]

$$= \frac{1}{3}\int_0^1 e^u du = \frac{1}{3}(e-1).$$ [1]

Hence $\int_0^1 \int_0^1 \int_{\sqrt{x}}^1 e^{y^3} dydxdz = \frac{1}{3}(e-1)$. [1]

(b) A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid sphere of radius 2 cms. Evaluate the volume cut out using double integral. Leave your answer with the integral expressions (i.e., there is no need to evaluate the integrals). [4]

Solution: $V = \int\int_R 2\sqrt{4 - x^2 - y^2} dA$ where $R = \{(x,y) : x^2 + y^2 \le 3\}$. [2]

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} 2\sqrt{4 - x^2 - y^2} dydx$$ [2]

OR $$= \int_0^{2\pi} \int_0^{\sqrt{3}} 2\sqrt{4 - r^2} rdrd\theta.$$

(c) Consider the surface $z = \sqrt{1 - x^2 - y^2}$ where $x^2 + y^2 \le 1$. Find the unit outward normal to the surface at the point $(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$. [2]

Solution: The unit outward normal is the same, i.e., $(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$. [2]

(7) (a) Let $f(x,y) = (xy^2 + sin^2 x)i + (x^2y + 2x + cos^2 y)j$. Find the line integral $\oint_C f \cdot dR$ where C is the triangle in the plane with the vertices $(0,0), (0,4)$ and $(1,1)$ and C is oriented counterclockwise. [4]

Solution: Let K denote the region bounded by the given triangle.
By Green's theorem $\oint_C f \cdot dR = \int\int_K 2\ dA$. [2]
$= 2$ (Area of the triangle)$= 4$. [2]

(b) Let C_1 be the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Let C_2 be the curve of intersection of the plane $y + z = 4$ and the cylinder $x^2 + y^2 = 4$. Suppose C_1 and C_2 are oriented counterclockwise when viewed from above. Let F be a vector field with $curlF = (1,0,1)$. Show that $\oint_{C_1} F \cdot dR = 4 \oint_{C_2} F \cdot dR$. [6]

Solution: If the surface S is defined by $z = f(x,y)$ then by the Stokes theorem $\oint_C F \cdot dR = \int\int_S (curlF) \cdot \widehat{n} d\sigma = \int\int_K (curlF) \cdot \frac{-f_x\hat{i} - f_y\hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}} \sqrt{f_x^2 + f_y^2 + 1} dxdy$. [2]
where K is the projection of the surface S on the xy-plane
Here the surfaces are portions from $z = 2 - y$ and $z = 4 - y$.
Hence $\oint_{C_1} F \cdot dR = \int\int_{x^2+y^2 \le 1} (1,0,1) \cdot (0,1,1) dxdy = \pi$. [2]
$\oint_{C_2} F \cdot dR = \int\int_{x^2+y^2 \le 4} (1,0,1) \cdot (0,1,1) dxdy = 4\pi$. [2]