1. (a) Show that \( \frac{\sin x}{x} = 2 \sin x - x \cos x \) cannot have 4 distinct real roots. [5]

**Solution:** Let \( f(x) = \frac{\sin x}{x} - 2 \sin x + x \cos x \). [1]
Suppose \( f \) has four zeros.
Then \( f'(x) = x^2 - x \sin x - \cos x \) has at least three zeros. [2]
So \( f''(x) = x(2 - \cos x) \) has at least two zeros. [1]
Note that \( f'' \) has only one zero which is a contradiction. [1]

(b) Let \( a_n > 0 \) and \( \sum_{n=1}^{\infty} a_n \) converges. Show that \( \sum_{n=1}^{\infty} (\frac{a_n}{n})^{1/2} \) converges. [5]

**Solution:** Observe that \( (\frac{a_n}{n})^{1/2} = \frac{\sqrt{a_n}}{\sqrt{n}} \leq \frac{1}{2} \left( a_n^{1/4} + \frac{1}{n^{1/4}} \right) \). [3]
Since \( a_n \to 0 \), \( a_n^{1/4} \leq a_n \), \( (\frac{a_n}{n})^{1/2} \leq \frac{1}{2} \left( a_n + \frac{1}{n} \right) \). [2]
(Or, \( \sum a_n^{1/4} \) converges by LCT with \( \sum a_n \) as \( a_n \to 0 \)).
By comparison test the series converges.
OR
\[
(\frac{a_n}{n})^{1/2} = \frac{\sqrt{a_n}}{\sqrt{n}} \leq \frac{1}{2} \left( a_n + \frac{1}{n} \right) a_n^{1/2} \leq \frac{1}{2} \left( a_n + \frac{1}{n} \right) \text{ as } a_n \to 0.
\]

(c) Let \( x_1 = 0, x_2 = 1 \) and \( x_{n+2} = \frac{1}{3} x_n + \frac{2}{3} x_{n+1} \) for \( n \geq 1 \). Show that the sequence \( (\sqrt{x_n} - \frac{1}{2}) \) converges but the series \( \sum_{n=1}^{\infty} (\sqrt{x_n} - \frac{1}{2}) \) does not converge. [6]

**Solution:** Note that \( |x_{n+2} - x_{n+1}| = \frac{1}{3} |x_{n} - x_{n+1}| \). [1]
By Cauchy Criterion \( (x_N) \) converges and hence \( (\sqrt{x_n} - \frac{1}{2}) \) converges. [1]
Observe that \( x_{n+2} + \frac{2}{3} x_{n+1} = \frac{1}{3} x_n + x_{n+1} \). [1]
If \( x_n \to \ell \), then \( \ell = \frac{3}{4} \). [2]
Since \( (\sqrt{x_n} - \frac{1}{2}) \to 0 \), \( \sum_{n=1}^{\infty} (\sqrt{x_n} - \frac{1}{2}) \) does not converge. [1]

2. (a) Determine all the values of \( x \) for which \( \sum_{n=1}^{\infty} \frac{(-1)^n n \sqrt{n(x-1)^n}}{n^{x+1}} \) converges. [5]

**Solution:** If for a fixed \( x \), \( a_n = \frac{(-1)^n n \sqrt{n(x-1)^n}}{n^{x+1}} \), then \( \left| \frac{a_{n+1}}{a_n} \right| \to |x - 1| \). [2]
Hence the power series converges for \( |x - 1| < 1 \). [1]
When \( x = 0 \), the series converges by LCT with \( \sum \frac{1}{n^{1/2}} \). [1]
When \( x = 2 \), the series converges because the series converges absolutely. [1]

(b) Let \( f : [-1, 1] \to \mathbb{R} \) be a twice differentiable function such that \( \int_0^1 f(t) dt = 0 \) for every \( n \in \mathbb{N} \). Show that \( f''(0) = 0 \). [6]

**Solution:** Let \( F(x) = \int_0^x f(t) dt \).
Then there exists \( c_n \in (0, \frac{1}{n}) \) s.t. \( F(\frac{1}{n}) - F(0) = f(c_n) \frac{1}{n} \Rightarrow f(c_n) = 0 \). [2]
By continuity of \( f \), \( f(c_n) \to f(0) \) and hence \( f(0) = 0 \). [1]
By differentiability of \( f \), \( f'(0) = \lim_{c_n \to 0} \frac{f(c_n) - f(0)}{c_n} = 0 \) \[1\]

By Rolle’s theorem, there exists \( d_n \in (0, c_n) \) s.t. \( f'(d_n) = 0 \) for every \( n \). \[1\]

Since \( f \) is twice differentiable, \( f''(0) = \lim_{d_n \to 0} \frac{f'(d_n) - f'(0)}{d_n} = 0 \) \[1\]

(c) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
n! & \text{if } n \leq x < n + \frac{1}{n} \text{ for every } n \in \mathbb{N} \\
0 & \text{otherwise}
\end{cases}
\]

(i) Does \( \int_0^\infty f(x)dx \) converge?

(ii) Does \( f(x) \to 0 \) as \( x \to \infty \)?

Justify your answer. \[6\]

**Solution:** Note that, since \( f(x) \geq 0 \),

\[
\int_0^\infty f(x)dx = \sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx
\]

\[= \sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx
\]

\[= \sum_{n=1}^{\infty} n! \frac{1}{n^n}
\]

Let \( a_n = n! \frac{1}{n^n} \), then \( \frac{a_{n+1}}{a_n} \to \frac{1}{e} \) \[1\]

Hence by Ratio Test the series converges. \[1\]

Hence the improper integral converges.

Since \( f(n) = n! \), \( f(x) \to 0 \) as \( x \to \infty \). \[1\]

3. (a) Let \( C \) be the curve defined by \( R(t) = (3 \cos t, t, 3 \sin t), t \geq 0 \).

i. Determine the curvature of \( C \).

ii. Find the position on the curve after traveling through a distance of \( \pi \sqrt{10} \) units along the curve from \( (3, 0, 0) \).

**Solution:** \( v(t) = (-3 \sin t, 1, 3 \cos t) \) and \( a(t) = (-3 \cos t, 0, -3 \sin t) \). \[1\]

Hence \( a(t) \times v(t) = 3 \sin t \cdot 9j - 3 \cos tk \).

Therefore, \( \kappa(t) = \frac{||a(t) \times v(t)||}{||v(t)||^3} = \frac{3}{10} \). \[2\]

Note that \( S(t) = \int_0^t \sqrt{10}du = \sqrt{10}t \).

Since \( \sqrt{10}t = \sqrt{10} \pi \), the position is \( (-3, \pi, 0) \). \[1\]

(b) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = \begin{cases} 
\frac{xy^2+y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases} \).

i. Determine the directions in which the directional derivatives of \( f \) exist at \( (0, 0) \).
ii. Determine whether \( f \) is differentiable at \((0, 0)\).

**Solution:** i. Let \((u, v) \in \mathbb{R}^2\) be such that \(\|(u, v)\| = 1\). Then
\[
\lim_{t \to 0} \frac{f(tu, tv) - f(0, 0)}{t} = \lim_{t \to 0} \frac{tu^2v^2 + t^9v^9}{t(t^2u^2 + t^9v^6)} \tag{1}
\]
\[
= \lim_{t \to 0} \frac{uv^2 + t^6v^9}{u^2 + t^6v^8} \tag{1}
\]

\[
= \begin{cases} 
 0 & \text{if } u = 0 \\
 2 & \text{if } u \neq 0 
\end{cases} \tag{1}
\]

ii. Note that \( f \) is not continuous at \((0, 0)\) as \(\lim_{x \to 0} f(x^4, x) = \lim_{x \to 0} \frac{x^6 + x^9}{x^3 + x^8} \neq 0\). \tag{2}

Since \( f \) is not continuous at \((0, 0)\), it is not differentiable at \((0, 0)\). \tag{1}

(c) Let \( S \) be the surface defined by \( z^2 = x^3 + y^2 \). Let \( P = (a, b, c) \in S \) where \( a \neq 0 \). Show that the tangent plane to the surface at \( P \) does not pass through the origin. \tag{5}

**Solution:** A normal at \( P \) is \((3a^2, 2b, -2c)\). \tag{1}

The equation of the tangent plane to \( S \) at \( P \) is
\[
3a^2x + 2by - 2cz - 3a^3 - 2b^2 + 2c^2 = 0. \tag{2}
\]

Since \( c^2 = a^3 + b^2 \), the equation becomes \(3a^2x + 2by - 2cz = a^3\). \tag{1}

Since \( a \neq 0 \), the plane does not pass through the origin. \tag{1}

4. (a) Let \( R \) be the triangular region bounded by the lines joining the points \((0, 0), (6, 0)\) and \((0, 4)\). Using a theorem of Pappus, find the coordinates of the centroid of this region. \tag{6}

**Solution:** Let the coordinates of the centroid be \((\bar{x}, \bar{y})\).

Revolve \( R \) about the y-axis:

By Pappus theorem \( V = 2\pi \bar{x}A = 2\pi \bar{x}12 = 48\pi \). \tag{2}

Hence \( \bar{x} = 2 \). \tag{1}

Revolve \( R \) about the x-axis:

By Pappus theorem \( V = 2\pi \bar{y}12 = 32\pi \). \tag{2}

Hence \( \bar{y} = \frac{4}{3} \). \tag{1}

(b) Find the area of the surface \( z = 2xy \) inside the cylinder \( x^2 + y^2 = 2 \). \tag{5}

**Solution:** Let \( f(x, y) = 2xy \). Then \( f_x = 2y \) and \( f_y = 2x \).

The required area
\[
A = \iint_{x^2+y^2<2} \sqrt{1 + 4(y^2 + x^2)} \, dxdy \tag{2}
\]

By polar co-ordinates,
\[
A = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4r^2} \, rdrd\theta. \tag{2}
\]

The value of \( A = \frac{13\pi}{3} \). \tag{1}

(c) Let \( C \) be the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( x + 2y + 2z = 3 \) which is oriented counterclockwise when viewed from the origin. Let \( F \) be a
vector field with \( \text{curl} F = \mathbf{i} + 2\mathbf{j} - \alpha \mathbf{k} \) for some \( \alpha \in \mathbb{R} \). Suppose \( \frac{g}{c} \mathbf{F} \cdot dR = \frac{-3\pi}{2} \).

Evaluate \( \alpha \).

**Solution:** By Stokes Theorem \( \frac{g}{c} \mathbf{F} \cdot dR = \iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} d\sigma \).

Note that \( \hat{n} = \pm \frac{i + 2j + 2k}{3} \) and in this case \( \hat{n} = \frac{-i - 2j - 2k}{3} \).

Hence \( \text{curl} \mathbf{F} \cdot \hat{n} = \frac{2\alpha - 5}{3} \).

Now \( \iint_S \frac{2\alpha - 5}{3} d\sigma = \iint_{x^2 + y^2 \leq 1} \frac{2\alpha - 3}{3} \sqrt{1 + f_x^2 + f_y^2} dxdy \)

where \( f(x, y) = x - 2y - \frac{5}{2} \).

Hence \( \iint_S \frac{2\alpha - 5}{3} d\sigma = \iint_{x^2 + y^2 \leq 1} \frac{2\alpha - 3}{3} 2dxdy = \frac{2\alpha - 5}{2} \pi \).

Therefore \( \alpha = 1 \).

5. (a) Let \( \int_0^1 e^{-x^2} dx = A \). Show that \( \int_0^1 \int_0^x e^{-t^2} dt dx = A + \frac{1}{2}(\frac{1}{e} - 1) \).

**Solution:** Note that

\[
\int_0^1 \int_0^x e^{-t^2} dt dx = \int_0^1 \int_t^1 e^{-x^2} dxdx
\]

\[
= A - \int_0^1 e^{-t^2} dt dx
\]

\[
= A - \frac{1}{2} \int_0^1 e^{-t} dt dx
\]

\[
= A + \frac{1}{2}(\frac{1}{e} - 1)
\]

(b) Let \( S_1 \) be the surface (of the upper hemisphere) \( x^2 + y^2 + z^2 = 1 \) with \( z > 0 \).

Let \( S_2 = \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\} \), the circular disk of radius 1 centered at \((0, 0, 0)\) in the plane \( z = 0 \). Evaluate

\[
\iint_{S_2} \{(z^2 \sin^2 z)x + y^2 + (e^x \sin x)z\} d\sigma - \iint_{S_2} \{(e^x \sin x)\} d\sigma.
\]

**Solution:** Let \( F(x, y, z) = (z^2 \sin^2 z)\mathbf{i} + y\mathbf{j} + (e^x \sin x)\mathbf{k} \).

The given surface integral \( \Upsilon = \iint_{S_1} F \cdot \mathbf{n}_1 d\sigma + \iint_{S_2} F \cdot \mathbf{n}_2 d\sigma \)

where \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) are the outward normals to \( S_1 \) and \( S_2 \) respectively.

By Divergence theorem \( \Upsilon = \iiint_D \text{div} F dV \)

where \( D \) is the solid upper hemisphere of radius 1 centered at \((0, 0, 0)\).

Since \( \text{div} F = 1 \), \( \Upsilon = \frac{2\pi}{3} \).

(c) Let \( f : [0, 1] \rightarrow (0, 1) \) be continuous. Show that the equation \( 2x - \int_0^x f(t) dt = 1 \) has exactly one solution in \([0, 1] \).

**Solution:** Let \( F(x) = 2x - \int_0^x f(t) dt - 1 \).

Then \( F'(x) = 2 - f(x) \neq 0 \) on \([0, 1] \) as \( f(x) \in (0, 1) \).

By Rolle’s Theorem \( F(x) = 0 \) can have at most one real root.

Note that \( F(0) = -1 \) and \( F(1) = 1 - \int_0^1 f(t) dt \geq 0 \) as \( f(x) \in (0, 1) \).

By IVP, \( F(x) = 0 \) has a solution in \([0, 1] \).
6.  (a) Let $D$ be the region bounded by the lines $x + y = 3$, $x + y = 6$, $x - y = 1$ and $x - y = 3$. Evaluate $\iint_D (x + y)^3 e^{2(x-y)} dxdy$. \[5\]

**Solution:** Let $u = x + y$ and $v = x - y$. Then $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$.

Then $|J(u, v)| = \frac{1}{2}$ \[1\]

\[
\iint_D (x + y)^3 e^{2(x-y)} dxdy = \frac{1}{2} \int_1^3 \int_3^6 u^3 e^{2v} dudv \]

\[
= \frac{1215}{8} \int_1^3 e^{2v} dv \]

\[
= \frac{1215}{16} (e^6 - e^2) \]

(b) Let $R$ be the triangle region bounded by the lines joining the points $(0, 0)$, $(2, 0)$ and $(2, 1)$. Let $f(x, y) = 2y^2 - 4y + x^2 - 4x + 1$. Find the points of maximum and minimum of $f$ on $R$. \[6\]

**Solution:** In the interior of $R$:

$f_x = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2$ and $f_y = 0 \Rightarrow 4y - 4 = 0 \Rightarrow y = 1$.

There is no critical point in the interior of $R$. \[2\]

Along the line joining $(0, 0)$ and $(2, 0)$: $f(x, 0) = x^2 - 4x + 1 = g(x)$.

$g'(x) = 0 \Rightarrow x = 2$. \[1\]

Along the line joining $(2, 0)$ and $(2, 1)$: $f(2, y) = 2y^2 - 4y - 3 = h(y)$.

$h'(y) = 0 \Rightarrow y = 1$. \[1\]

Along the line joining $(0, 0)$ and $(2, 1)$: $f(x, \frac{x}{2}) = \frac{3}{2}x^2 - 6x + 1 = k(x)$.

$k'(x) = 0 \Rightarrow x = 2$. \[1\]

Therefore maximum and minimum can occur only at $(0, 0)$, $(2, 0)$ and $(2, 1)$.

Note that $f(0, 0) = 1$, $f(2, 1) = -5$ and $f(2, 0) = -3$.

Hence point of maximum is $(0, 0)$ and point of minimum is $(2, 1)$. \[1\]

(c) Evaluate $\oint_C 2xyz dx + x^2 zdy + x^2 ydz$ where $C$ is the parametrized curve $R(t) = \cos t \hat{i} + \frac{t}{2\pi} \hat{j} + \sin t \hat{k}$, $0 \leq t \leq \frac{\pi}{2}$. \[5\]

**Solution:** Observe that

$\nabla \phi = (2xyz, x^2 z, x^2 y)$ where $\phi(x, y, z) = x^2 yz$.

So, $\oint_C 2xyz dx + x^2 zdy + x^2 ydz = \oint_C \nabla \phi \cdot dR = \phi(R(\frac{\pi}{2})) - \phi(R(0))$. \[1\]

Note that $R(0) = (1, 0, 0)$ and $R(\frac{\pi}{2}) = (0, \frac{1}{4}, 1)$.

Therefore, $\oint_C 2xyz dx + x^2 zdy + x^2 ydz = 0$. \[1\]