MTH101AA (2020), Tentative Marking Scheme - End sem. exam

1. (a) Show that $\frac{x^3}{3} = 2\sin x - x\cos x$ cannot have 4 distinct real roots. [5]

   Solution: Let $f(x) = \frac{x^3}{3} - 2\sin x + x\cos x$. [1]
   Suppose f has four zeros.
   Then $f'(x) = x^2 - x\sin x - \cos x$ has at least three zeros. [2]
   So $f''(x) = x(2 - \cos x)$ has at least two zeros. [1]
   Note that f'' has only one zero which is a contradiction. [1]

   (b) Let $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} (\frac{a_n}{n})^{\frac{1}{2}+\frac{1}{5}}$ converges. [5]

   Solution: Observe that $(\frac{a_n}{n})^{\frac{1}{2}+\frac{1}{5}} = \frac{\sqrt{a_n^{\frac{14}{10}}}}{\sqrt{n^{\frac{14}{10}}}} \leq \frac{1}{2}\left(a_n^{\frac{14}{10}} + \frac{1}{n^{\frac{14}{10}}}\right)$. [3]
   Since $a_n \to 0$, $a_n^{\frac{14}{10}} \leq a_n$, $(\frac{a_n}{n})^{\frac{1}{2}+\frac{1}{5}} \leq \frac{1}{2}\left(a_n + \frac{1}{n^{\frac{14}{10}}}\right)$. [2]
   (Or, $\sum a_n^{\frac{14}{10}}$ converges by LCT with $\sum a_n$ as $a_n \to 0$).
   By comparison test the series converges.
   OR
   $(\frac{a_n}{n})^{\frac{1}{2}+\frac{1}{5}} = \frac{\sqrt{a_n}a_n^{\frac{1}{5}}}{\sqrt{n^{\frac{14}{10}}}} \leq \frac{1}{2}\left(a_n + \frac{1}{n^{\frac{14}{10}}}\right)a_n^{\frac{1}{5}} \leq \frac{1}{2}\left(a_n + \frac{1}{n^{\frac{14}{10}}}\right)$ as $a_n \to 0$.

   (c) Let $x_1 = 0, x_2 = 1$ and $x_{n+2} = \frac{1}{3}x_n + \frac{2}{3}x_{n+1}$ for $n \geq 1$. Show that the sequence $\left(\sqrt{x_n} - \frac{1}{2}\right)$ converges but the series $\sum_{n=1}^{\infty}\left(\sqrt{x_n} - \frac{1}{2}\right)$ does not converge. [6]

   Solution: Note that $|x_{n+2} - x_{n+1}| = \frac{1}{3}|x_n - x_{n+1}|$. [1]
   By Cauchy Criterion (x_N) converges and hence $\left(\sqrt{x_n} - \frac{1}{2}\right)$ converges. [1]
   Observe that $x_{n+2} + \frac{1}{3}x_{n+1} = \frac{1}{3}x_n + x_{n+1}$. [1]
   If $x_n \to \ell$, then $\ell = \frac{3}{4}$ [2]
   Since $\left(\sqrt{x_n} - \frac{1}{2}\right) \nrightarrow 0$, $\sum_{n=1}^{\infty}\left(\sqrt{x_n} - \frac{1}{2}\right)$ does not converge. [1]

2. (a) Determine all the values of x for which $\sum_{n=1}^{\infty} \frac{(-1)^n n\sqrt{n}(x-1)^n}{n^3+1}$ converges. [5]

   Solution: If for a fixed x, $a_n = \frac{(-1)^n n\sqrt{n}(x-1)^n}{n^3+1}$, then $\left|\frac{a_{n+1}}{a_n}\right| \to |x-1|$. [2]
   Hence the power series converges for $|x-1| < 1$. [1]
   When $x = 0$, the series converges by LCT with $\sum \frac{1}{n^{\frac{3}{2}}}$ [1]
   When $x = 2$, the series converges because the series converges absolutely. [1]

   (b) Let $f : [-1, 1] \to \mathbb{R}$ be a twice differentiable function such that $\int_0^{1/n} f(t)dt = 0$ for every $n \in \mathbb{N}$. Show that $f''(0) = 0$. [6]

   Solution: Let $F(x) = \int_0^x f(t)dt$.
   Then there exists $c_n \in (0, \frac{1}{n})$ s.t. $F(\frac{1}{n}) - F(0) = f(c_n)\frac{1}{n} \Rightarrow f(c_n) = 0$ [2]
   By continuity of f, $f(c_n) \to f(0)$ and hence $f(0) = 0$. [1]

By differentiability of f, $f'(0) = \lim_{c_n \to 0} \frac{f(c_n)-f(0)}{c_n} = 0$ [1]

By Rolle's theorem, there exists $d_n \in (0, c_n)$ s.t. $f'(d_n) = 0$ for every n. [1]

Since f is twice differentiable, $f''(0) = \lim_{d_n \to 0} \frac{f'(d_n)-f'(0)}{d_n} = 0$ [1]

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} n! & if\ n \le x < n + \frac{1}{n^n} \text{ for every } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

(i) Does $\int_0^\infty f(x)dx$ converge?

(ii) Does $f(x) \to 0$ as $x \to \infty$?

Justify your answer. [6]

Solution: Note that, since $f(x) \ge 0$,

$$\begin{aligned} \int_0^\infty f(x)dx &= \sum_{n=1}^\infty \int_n^{n+1} f(x)dx && [1] \\ &= \sum_{n=1}^\infty \int_n^{n+\frac{1}{n^n}} f(x)dx && [1] \\ &= \sum_{n=1}^\infty n!\frac{1}{n^n} && [1] \end{aligned}$$

Let $a_n = n!\frac{1}{n^n}$, then $\frac{a_{n+1}}{a_n} \to \frac{1}{e}$ [1]

Hence by Ratio Test the series converges. [1]

Hence the improper integral converges.

Since $f(n) = n!$, $f(x) \nrightarrow 0$ as $x \to \infty$. [1]

3. (a) Let C be the curve defined by $R(t) = (3\cos t, t, 3\sin t), t \ge 0$.

i. Determine the curvature of C.

ii. Find the position on the curve after traveling through a distance of $\pi\sqrt{10}$ units along the curve from $(3, 0, 0)$. [6]

Solution: $v(t) = (-3\sin t, 1, 3\cos t)$ and $a(t) = (-3\cos t, 0, -3\sin t)$. [1]

Hence $a(t) \times v(t) = 3\sin t i + 9j - 3\cos t k$. [1]

Therefore, $\kappa(t) = \frac{\|a(t)\times v(t)\|}{\|v(t)\|^3} = \frac{3}{10}$. [2]

Note that $S(t) = \int_0^t \sqrt{10}du = \sqrt{10}t$. [1]

Since $\sqrt{10}t = \sqrt{10}\pi$, the position is $(-3, \pi, 0)$. [1]

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy^2+y^9}{x^2+y^8} & if (x,y) \ne (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. [6]

i. Determine the directions in which the directional derivatives of f exist at $(0,0)$.

ii. Determine whether f is differentiable at $(0,0)$.

Solution: i. Let $(u,v) \in \mathbb{R}^2$ be such that $\|(u,v)\| = 1$. Then

$$\begin{aligned} \lim_{t\to 0} \frac{f(tu,tv)-f(0,0)}{t} &= \lim_{t\to 0} \frac{tut^2v^2+t^9v^9}{t(t^2u^2+t^8v^8)} & [1] \\ &= \lim_{t\to 0} \frac{uv^2+t^6v^9}{u^2+t^6v^8} & [1] \\ &= \begin{cases} v & \text{if } u=0 \\ \frac{v^2}{u} & \text{if } u\neq 0 \end{cases} & [1] \end{aligned}$$

ii. Note that f is not contnuous at $(0,0)$ as $\lim_{x\to 0} f(x^4,x) = \lim_{x\to 0} \frac{x^6+x^9}{x^8+x^8} \neq 0$. [2]
Since f is not continuous at $(0,0)$, it not differentiable at $(0,0)$. [1]

(c) Let S be the surface defined by $z^2 = x^3 + y^2$. Let $P = (a,b,c) \in S$ where $a \neq 0$. Show that the tangent plane to the surface at P does not pass through the origin. [5]

Solution: A normal at P is $(3a^2, 2b, -2c)$. [1]
The equation of the tangent plane to S at P is
$3a^2(x-a) + 2b(y-b) - 2c(z-c) = 0$ [2]
i.e., $3a^2x + 2by - 2cz - 3a^3 - 2b^2 + 2c^2 = 0$.
Since $c^2 = a^3 + b^2$, the equation becomes $3a^2x + 2by - 2cz = a^3$. [1]
Since $a \neq 0$, the plane does not pass through the origin. [1]

4. (a) Let R be the triangular region bounded by the lines joining the points $(0,0)$, $(6,0)$ and $(0,4)$. Using a theorem of Pappus, find the coordinates of the centroid of this region. [6]

Solution: Let the coordinates of the centroid be $(\overline{x}, \overline{y})$.
Revolve R about the y-axis:
By Pappus theorem $V = 2\pi\overline{x}A = 2\pi\overline{x}12 = 48\pi$. [2]
Hence $\overline{x} = 2$. [1]
Revolve R about the x-axis:
By Pappus theorem $V = 2\pi\overline{y}12 = 32\pi$. [2]
Hence $\overline{y} = \frac{4}{3}$. [1]

(b) Find the area of the surface $z = 2xy$ inside the cylinder $x^2 + y^2 = 2$. [5]

Solution: Let $f(x,y) = 2xy$. Then $f_x = 2y$ and $f_y = 2x$.
The required area $A = \iint_{x^2+y^2\leq 2} \sqrt{1+4(y^2+x^2)}dxdy$ [2]
By polar co-ordinates, $A = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}rdrd\theta$. [2]
The value of $A = \frac{13\pi}{3}$ [1]

(c) Let C be the intersection of the cylinder $x^2+y^2 = 1$ with the plane $x+2y+2z = 3$ which is oriented counterclockwise when viewed from the origin. Let F be a

vector field with $curlF = \widehat{i}+2\widehat{j}-\alpha\widehat{k}$ for some $\alpha \in \mathbb{R}$. Suppose $\oint\limits_C F \cdot dR = \frac{-3\pi}{2}$.
Evaluate α. [6]

Solution: By Stokes Theorem $\oint\limits_C F \cdot dR = \iint_S curlF \cdot \widehat{n} d\sigma$. [1]

Note that $\widehat{n} = \pm\frac{i+2j+2k}{3}$ and in this case $\widehat{n} = \frac{-i-2j-2k}{3}$. [1]

Hence $curlF \cdot \widehat{n} = \frac{2\alpha-5}{3}$. [1]

Now $\iint_S \frac{2\alpha-5}{3} d\sigma = \iint_{x^2+y^2\leq 1} \frac{2\alpha-3}{3}\sqrt{1+f_x^2+f_y^2}dxdy$ [1]

where $f(x,y) = \frac{3}{2} - y - \frac{x}{2}$.

Hence $\iint_S \frac{2\alpha-5}{3} d\sigma = \iint_{x^2+y^2\leq 1} \frac{2\alpha-3}{3}\frac{3}{2}dxdy = \frac{2\alpha-5}{2}\pi$. [1]

Therefore $\alpha = 1$. [1]

5. (a) Let $\int\limits_0^1 e^{-x^2}dx = A$. Show that $\int_0^1\int_0^x e^{-t^2}dtdx = A + \frac{1}{2}(\frac{1}{e}-1)$. [5]

Solution: Note that

$$\begin{aligned}\int_0^1\int_0^x e^{-t^2}dtdx &= \int_0^1\int_t^1 e^{-t^2}dxdt && [3]\\ &= A - \int_0^1 e^{-t^2}tdt && [1]\\ &= A - \frac{1}{2}\int_0^1 e^{-t}dt && [1]\\ &= A + \frac{1}{2}(\frac{1}{e}-1)\end{aligned}$$

(b) Let S_1 be the surface (of the upper hemisphere) $x^2+y^2+z^2 = 1$ with $z > 0$. Let $S_2 = \{(x,y,z) : x^2+y^2 \leq 1, z = 0\}$, the circular disk of radius 1 centered at $(0,0,0)$ in the plane $z = 0$. Evaluate
$\iint_{S_1}\{(z^2\sin^2 z)x + y^2 + (e^x \sin x)z\}d\sigma - \iint_{S_2}\{(e^x\sin x)\}d\sigma$. [6]

Solution: Let $F(x,y,z) = (z^2\sin^2 z)\widehat{i} + y\widehat{j} + (e^x\sin x)\widehat{k}$. [1]

The given surface integral $\Upsilon = \iint_{S_1} F\cdot\widehat{n_1}d\sigma + \iint_{S_2} F\cdot\widehat{n_2}d\sigma$ [2]

where $\widehat{n_1}$ and $\widehat{n_2}$ are the outward normals to S_1 and S_2 respectively.

By Divergence theorem $\Upsilon = \iiint_D divFdV$ [2]

where D is the solid upper hemisphere of radius 1 centered at $(0,0,0)$.

Since $divF = 1$, $\Upsilon = \frac{2\pi}{3}$ [1]

(c) Let $f : [0,1] \to (0,1)$ be continuous. Show that the equation $2x - \int_0^x f(t)dt = 1$ has exactly one solution in $[0,1]$. [6]

Solution: Let $F(x) = 2x - \int_0^x f(t)dt - 1$. [1]

Then $F'(x) = 2 - f(x) \neq 0$ on $[0,1]$ as $f(x) \in (0,1)$. [2]

By Rolle's Theorem $F(x) = 0$ can have at most one real root. [1]

Note that $F(0) = -1$ and $F(1) = 1 - \int_0^1 f(t)dt \geq 0$ as $f(x) \in (0,1)$. [1]

By IVP, $F(x) = 0$ has a solution in $[0,1]$. [1]

6. (a) Let D be the region bounded by the lines $x+y=3, x+y=6, x-y=1$ and $x-y=3$. Evaluate $\iint_D (x+y)^3 e^{2(x-y)} dxdy$. [5]

Solution: Let $u = x+y$ and $v = x-y$. Then $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$.
Then $|J(u,v)| = \frac{1}{2}$ [1]

$$\iint_D (x+y)^3 e^{2(x-y)} dxdy = \frac{1}{2}\int_1^3 \int_3^6 u^3 e^{2v} du dv \quad [2]$$

$$= \frac{1215}{8} \int_1^3 e^{2v} dv \quad [1]$$

$$= \frac{1215}{16}(e^6 - e^2) \quad [1]$$

(b) Let R be the triangle region bounded by the lines joining the points $(0,0), (2,0)$ and $(2,1)$. Let $f(x,y) = 2y^2 - 4y + x^2 - 4x + 1$. Find the points of maximum and minimum of f on R. [6]

Solution: In the interior of R:
$f_x = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2$ and $f_y = 0 \Rightarrow 4y - 4 = 0 \Rightarrow y = 1$.
There is no critical point in the interior of R. [2]
Along the line joining $(0,0)$ and $(2,0)$: $f(x,0) = x^2 - 4x + 1 = g(x)$.
$g'(x) = 0 \Rightarrow x = 2$. [1]
Along the line joining $(2,0)$ and $(2,1)$: $f(2,y) = 2y^2 - 4y - 3 = h(y)$.
$h'(y) = 0 \Rightarrow y = 1$. [1]
Along the line joining $(0,0)$ and $(2,1)$: $f(x, \frac{x}{2}) = \frac{3}{2}x^2 - 6x + 1 = k(x)$.
$k'(x) = 0 \Rightarrow x = 2$. [1]
Therefore maximum and minimum can occur only at $(0,0), (2,0)$ and $(2,1)$.
Note that $f(0,0) = 1, f(2,1) = -5$ and $f(2,0) = -3$.
Hence point of maximum is $(0,0)$ and point of minimum is $(2,1)$. [1]

(c) Evaluate $\oint_C 2xyzdx + x^2zdy + x^2ydz$ where C is the parametrized curve $R(t) = \cos t\ \widehat{i} + \frac{t}{2\pi}\ \widehat{j} + \sin t\ \widehat{k}, 0 \le t \le \frac{\pi}{2}$. [5]

Solution: Observe that
$\nabla\phi = (2xyz, x^2z, x^2y)$ where $\phi(x,y,z) = x^2yz$. [2]
So, $\oint_C 2xyzdx + x^2zdy + x^2ydz = \oint_C \nabla\phi \cdot dR = \phi(R(\frac{\pi}{2})) - \phi(R(0))$. [1]
Note that $R(0) = (1,0,0)$ and $R(\frac{\pi}{2}) = (0, \frac{1}{4}, 1)$. [1]
Therefore, $\oint_C 2xyzdx + x^2zdy + x^2ydz = 0$. [1]