Tentative marking scheme for End-Sem. Exam: MTH 101N, 2008-09 (I)

1 (a) $\dfrac{x_n + (1-x_{n+1})}{2} \geq \sqrt{x_n(1-x_{n+1})} > \frac{1}{2}$ .... (2)

$\Rightarrow x_n - x_{n+1} > 0$, ie (x_n) is monotone and hence it converges .... (2)

Let $x_n \to x_0$. Since $4x_n(1-x_n) > 1$, $4x_0(1-x_0) \geq 1$ ie $(2x_0-1)^2 \leq 0$. Hence $x_0 = \frac{1}{2}$ ... (2)

(b) Note that $f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$ ... (2)

Let $M > 0$ be such that $M > f(0)$ (or $f(\alpha)$ for any fixed α) .... (1)

Then $\exists\, p$ s.t. $f(x) > M$ for all x such that $|x| > p$ .... (2)

Since f is continuous $\exists\, x_0$ s.t $f(x_0) = \inf\limits_{x \in [-p,p]} f(x) = \inf\limits_{x \in \mathbb{R}} f(x)$ .... (1)

2(a) $\lim\limits_{n\to\infty} \left(\dfrac{(n+1)^m + (n+2)^m + \cdots + (n+k)^m}{n^{m-1}} - kn \right) = \lim\limits_{n\to\infty} \dfrac{(n+1)^m - n^m + (n+2)^m - n^m + \cdots + (n+k)^m - n^m}{n^{m-1}}$ ....(2)

$= \lim\limits_{n\to\infty} \left(\dfrac{(1+\frac{1}{n})^m - 1}{\frac{1}{n}} + 2\, \dfrac{(1+\frac{2}{n})^m - 1}{\frac{2}{n}} + \cdots + k\, \dfrac{(1+\frac{k}{n})^m - 1}{\frac{k}{n}} \right)$ ....(2)

$= m + 2m + \cdots + km$ ----(1)

(b) By MVT, $\exists\, c_1 \in (0, \frac{1}{4})$ s.t $f'(c_1) = \dfrac{f(\frac{1}{4}) - f(0)}{\frac{1}{4}} = 4$ .... (2)

By MVT, $\exists\, c_2 \in (\frac{3}{4}, 1)$ s.t $f'(c_2) = \dfrac{f(3/4) - f(1)}{-\frac{1}{4}} = -4$ ... (2)

By Rolles thm. $\exists\, c_3 \in (\frac{1}{4}, \frac{3}{4})$ s.t $f'(c_3) = 0$ .... (2).

3 (a) By Ratio test, the series converges for $\left|2\frac{x+1}{x}\right| < 1$ .... (2)

The series converges on $(-1, -1/3)$ .... (1)

The series converges for $x = -1/3$ and diverges for $x = -1$ ....(2)

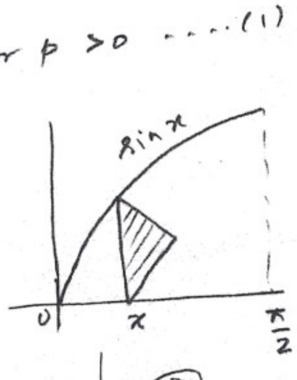
(b) Since $\int_1^\infty \left|\dfrac{\sin x}{x^p}\right| dx \leq \int_1^\infty \dfrac{dx}{x^p}$, the integral converges abs. for $p > 1$ ...(1)

By Dirichlet test, $\int_1^\infty \dfrac{\sin x}{x^p}\,dx$ converges for $0 < p$. .... (1)

Let $0 < p \leq 1$. Note that $\left|\dfrac{\sin x}{x^p}\right| \geq \dfrac{\sin^2 x}{x^p} \geq \dfrac{1-\cos 2x}{2x^p}$ .... (2)

Since $\int_1^\infty \dfrac{1}{2x^p}$ diverges for $p \leq 1$ ---(1) and $\int_1^\infty \dfrac{\cos 2x}{x^p}$ conv. for $p > 0$ ....(1)

$\int_1^\infty \dfrac{\sin x}{x^p}$ diverges for $0 < p \leq 1$. ---- (1)

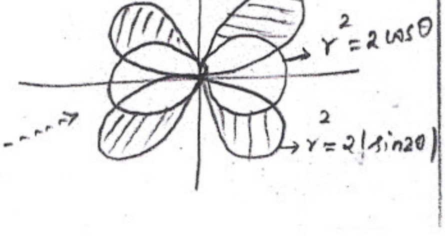


4 (a) $V = \int_a^b A(x)\,dx$ ... (1). For $x \in [0, \pi/2]$, $A(x) = \frac{1}{2}\sin^2 x \cdot \frac{\sqrt{3}}{2}$ ....(2)

So $V = \dfrac{\sqrt{3}}{4} \int_0^{\pi/2} \sin^2 x\,dx = \dfrac{\sqrt{3}}{16}\pi$ ... (1) ... (1)

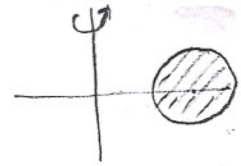
(b) $2\sin 2\theta = 2\cos\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ .... (1)

(1+1)



$A_1 = \frac{1}{2}\int_{\pi/6}^{\pi/2} (2\sin 2\theta - 2\cos\theta)\, d\theta$ .... (1). Required area $= 4A_1$ .... (1).

4 (c). By Pappus thm, volume $= 2\pi \rho A = 2\pi\cdot 4\cdot \pi 4$ .... (2)



5 (a). Curvature $K = \frac{2}{(1+(4-2x)^2)^{3/2}}$ .... (2)

K is max. when $(4-2x)^2$ is minimum. .... (1)

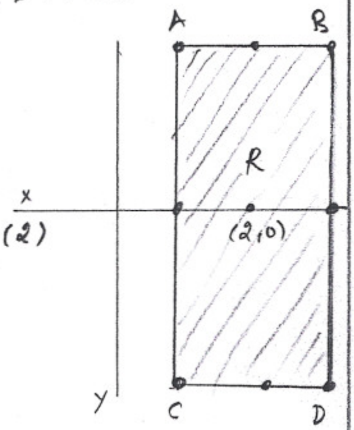
$\frac{d}{dx}((4-2x)^2) = 0 \Rightarrow x = 2$ .... (1). The curv. is max. at $(2,4)$ .... (1)

(b) For $\|(u_1,u_2)\|=1$, $\lim_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \begin{cases} 0 & \text{if } u_2 = 0 \quad \ldots (1)\\ \frac{u_2}{|u_2|} & \text{if } u_2\neq 0 \quad \ldots (1)\end{cases}$

Note that $f_x(0,0)=0$ and $f_y(0,0)=1$ .... (1). Note that

$$\epsilon(h,k) = \frac{f(0+(h,k)) - f(0,0) - (0,1)\cdot(h,k)}{\|(h,k)\|} = \frac{\frac{k}{|k|}\sqrt{h^2+k^2} - k}{\sqrt{h^2+k^2}} \quad \ldots (2)$$

For $h = k$, $\epsilon(h,k) = \left(\frac{\sqrt2 - 1}{\sqrt2}\right)\frac{k}{|k|} \not\to 0$ as $k\to 0$ .... (2).



6(a). • $f_x = 0 \Rightarrow (2x-4)\cos y = 0 \Rightarrow x = 2$

$f_y = 0 \Rightarrow (x^2-4x)(-\sin y) = 0 \Rightarrow x=4$ or $y=0$.

• $(2,0)$ is a critical pt in the interior of R & $f(2,0) = -4$ .... (2)

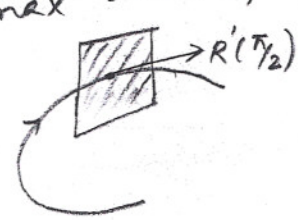
• $\left.\begin{array}{l}\text{Along } AC: f(1,y) = -3\cos y \\ \text{Along } BD: f(3,y) = -3\cos y\end{array}\right\} = g(y)$

$g'(y) = 3\sin y = 0 \Rightarrow y = 0$ & $f(1,0) = -3$ & $f(3,0) = -3$ .... (2)

• $\left.\begin{array}{l}\text{Along } CD: f(x,-\pi/4) = \frac{x^2-4x}{\sqrt2} \\ \text{Along } AB: f(x,\pi/4) = \frac{x^2-4x}{\sqrt2}\end{array}\right\} = h(x)$

$h'(x) = \frac{2x-4}{\sqrt2} = 0 \Rightarrow x=2$ & $f(2,\frac{\pi}{4}) = f(2,-\frac{\pi}{4}) = \frac{-4}{\sqrt2}$ .... (2)

• $f(1,\frac{\pi}{4}) = f(1,-\pi/4) = f(3,\pi/4) = f(3,-\pi/4) = -\frac{3}{2}$ → max .... (2)



• $f(2,0) = -4$ → min .... (1).

b) The equation of the plane is $R'(\frac{\pi}{2})\cdot(x,y,z) = R'(\pi/2)\cdot R(\frac{\pi}{2})$ .... (2)

7(a) Wire is cut into two parts whose lengths are x and y.

• Sum of the areas is $\frac{x^2}{16} + \pi\left(\frac{y}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{y^2}{4\pi} = f(x,y)$ .... (2)

Subject to $g(x,y) = x+y-b=0$

- By Lag. Mult. method : $\nabla f = \lambda \nabla g \Rightarrow \frac{x}{8} = \lambda$ & $\frac{y}{2\pi} = \lambda$ ....(1)

$\Rightarrow 8\lambda + 2\pi\lambda - b = 0$ i.e. $\lambda(8+2\pi) = b$ ....(2)

- We obtain the points $\bar{x} = \frac{8b}{8+2\pi}$ and $\bar{y} = \frac{b.2\pi}{8+2\pi}$ ....(2)

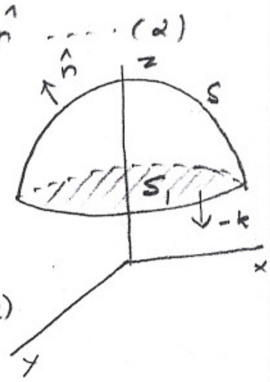
- The least value is $\frac{64b^2}{16(8+2\pi)^2} + \frac{4\pi^2 b^2}{(8+2\pi)^2}$ ....(1)

7(b) $\int_0^1 \left(\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx\right) dz = \int_0^1 \left(\int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy\right) dz = \int_0^1 \int_0^\pi \sin y\, dy\, dz = 2$ ....(1)

(3)

8(a) The unit outer normal to S is $\hat{n} = \left(\frac{x}{3}, \frac{y}{3}, \frac{z-1}{3}\right)$ ....(1)

Note that $\frac{x^2}{6} + \frac{y^2}{6} + \frac{(z-1)^2}{3} = \left(\frac{x}{2}, \frac{3y}{2}, z-1\right) \cdot \left(\frac{x}{3}, \frac{y}{3}, \frac{z-1}{3}\right) = F.\hat{n}$ ....(2)



Let $S_1 := (x, y, z)$, $z = 1$ & $x^2 + y^2 \le 3$. Note that $\operatorname{div} F = 3$ ..(1)

By div. Thm. $\iint_S \cdots d\sigma = \iiint_D \operatorname{div} F - \iint_{S_1} F.(-k)\, d\sigma$ ...(2)

$= 3 \cdot \frac{2}{3}\pi 3^{3} - 0$ as $F.k = z - 1 = 0$ on S_1 ...(2)

(b) $\iiint_D (x^2+y^2)\, dv = \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho^2 \sin^2\phi \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ ...(2)

$= \frac{b^5 - a^5}{5} \int_0^{2\pi} \int_0^{\pi} (1 - \cos^2\varphi) \sin\varphi\, d\varphi\, d\theta$

$= \frac{b^5 - a^5}{5} \int_0^{2\pi} \left[-\cos\varphi + \frac{1}{3}\cos^3\varphi\right]_0^\pi d\theta = \frac{4}{15}(b^5 - a^5) \int_0^{2\pi} d\theta = \frac{8\pi}{15}(b^5 - a^5)$ ...(1+1)

9 Unit outer normal $\hat{n} = \frac{2xi + k}{\sqrt{1+4x^2}}$ ......(1)

Curl $F = -k$ ....(1)

By Stokes thm $\oint_C (yi + yj + zk)\, dR = \iint_S \operatorname{curl} F.\hat{n}\, d\sigma$ ....(1)

$\operatorname{curl} F.\hat{n} = \frac{-1}{\sqrt{1+4x^2}}$ ...(1)

So $\iint_S \operatorname{curl} F.\hat{n}\, d\sigma = \iint_S \frac{-1}{\sqrt{1+4x^2}}\, d\sigma = \int_0^1 \int_{-2}^{2} -1\, dy\, dx$ ....(2)