Indian Institute Of Technology Kanpur

FIRST MIDSEM EXAM, MTH-101

Date: 17-09-2014

Duration: 2 Hour

1 (a) Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n}$ for $n \in \mathbb{N}$. Show that (x_n) satisfies Cauchy criterion. Find the limit of (x_n). [6]

Solution. Note that

$$|x_{n+2} - x_{n+1}| = \frac{1}{(2+x_{n+1})(2+x_n)}|x_n - x_{n+1}| < \frac{1}{4}|x_{n+1} - x_n|.$$

Therefore (x_n) satisfies the contractive condition with $\alpha = 1/4$, and hence it satisfies the Cauchy criterion. [3]

If $x_n \to l$ then $l = \frac{1}{2+l}$, which gives $l^2 + 2l - 1 = 0$. Hence $(l+1)^2 = 2$, that is $l + 1 = \pm\sqrt{2}$ or $l = 1 \pm \sqrt{2}$. [2]

Since $x_n > 0$ for all n, $l = -1 + \sqrt{2}$. [1]

(b) Suppose the sequence $(\cos nx)$ is convergent for some $x \in \mathbb{R}$. Show that $x = 2k\pi$ for some integer k. [6]

Solution. Suppose that $a_n := \cos nx \to l$. Since

$$\cos(n+1)x + \cos(n-1)x = 2\cos nx \cos x,$$

by taking limit, we get $2l = 2l\cos x$. Thus $l = 0$ or $\cos x = 1$. [3]

Since $\cos 2nx = 2\cos^2 nx - 1$, by taking limit, we get $l = 2l^2 - 1$. Hence $l \neq 0$. Hence $\cos x = 1$, that is, $x = 2k\pi$ for some k. [3]

Note: Give full marks if students use some other trigonometric identities to get the conclusion.

2 (a).i Let $p(x) : \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. Show that $p(x)$ is an onto function. [3+3=6]

Solution. Suppose $p(x) = a_0 + a_1 x + \cdots + a_n x^{2n+1}$ and let $\lambda \in \mathbb{R}$. If $a_n > 0$ then $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to -\infty$ as $x \to -\infty$. If $a_n < 0$ then $p(x) \to -\infty$ as $x \to \infty$ and $p(x) \to \infty$ as $x \to -\infty$. Thus there exists x_1, x_2 such that $p(x_1) \leq \lambda \leq p(x_2)$. By IVP, there exists x_3 such that $p(x_3) = \lambda$. [3]

Alternative Solution. By IVP, every odd degree polynomial has a real root. Applying this to $p(x) - \lambda$ for $\lambda \in \mathbb{R}$, we get x_0 such that $p(x_0) = \lambda$. [3]

ii Show that there is no continuous function which maps $[0, 1]$ onto $(0, 1)$.

Solution. Suppose f is continuous such $f([0, 1]) = (0, 1)$. In particular, $1 \notin f([0, 1])$. Note that $\sup_{x\in[0,1]} f(x) = \sup(0, 1) = 1$. Since f is continuous, $1 = \sup_{x\in[0,1]} f(x) = f(x_0)$ for some $x_0 \in [0, 1]$, which is not possible by our assumption. [3]

Alternative Solution. If f is continuous then $f([0, 1]) = [\inf f(x), \sup f(x)]$. But $\inf_{x\in[0,1]} f(x) = 0$ and $\sup_{x\in[0,1]} f(x) = 1$. Thus we obtain $f([0, 1]) = [0, 1]$, which is a contradiction. [3]

(b) Let $p(x) = a + bx + cx^2$ be a quadratic polynomial. Find all values of $a, b, c \in \mathbb{R}$ for which the function $p(|x|)$ is differentiable at 0. [6]

Solution. Note that [3]

$$\lim_{h\to 0} \frac{p(|h|) - p(0)}{h} = \lim_{h\to 0} \frac{a + b|h| + c|h|^2 - a}{h} = b \lim_{h\to 0} \frac{|h|}{h}.$$

If $b = 0$ then clearly the derivative exists at 0. If $b \neq 0$ then by choosing paths $h > 0$ and $h < 0$, one can see that $\lim_{h\to 0} \frac{p(|h|)-p(0)}{h}$ takes values b and $-b$ respectively, so derivative does not exist. [3]

Thus $p(|x|)$ is differentiable at 0 for $b = 0$ and all values a, c.

3 (a) A cylindrical box is to be made. The volume V of the box should be 250π cm. Find the height and radius of the box that minimizes the amount of material to be used. Recall: The volume V and surface area A are given by $V = \pi r^2 h$ and $A = 2\pi r^2 + 2\pi r h$ respectively. [6]

Solution. The volume is given by $V = \pi r^2 h = 250\pi$. Thus $h = \frac{V}{\pi r^2} = \frac{250}{r^2}$. The surface area is given by $A = 2\pi r^2 + 2\pi r h$, where h is height and r is the radius of the can. This gives $A = 2\pi r^2 + \frac{500}{r}\pi$. [3]

Note that $\frac{dA}{dr} = 4\pi r - \frac{500}{r^2}\pi$ is negative if $r < 5$ and positive if $r > 5$. Thus A has minimum when $r = 5$. It follows that $h = 10$. [3]

(b) Find intervals of concavity/convexity and points of local minima/maxima of $f(x) = \frac{x^2}{x^2-1}$ for $x \neq 1$. [6]

Solution. Note that $f(x) = 1 + \frac{1}{x^2-1}$. Thus $f'(x) = \frac{-2x}{(x^2-1)^2}$ and $f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$. [1]

Fact: If $f''(x) > 0$ (resp. $f''(x) < 0$) for all $x \in (a, b)$ then f is convex (resp. concave). Note that $f'' > 0$ on $(-\infty, -1)$ and $(1, \infty)$. Thus f is convex on $(-\infty, -1)$ and $(1, \infty)$. Note that $f'' < 0$ on $(-1, 1)$. Thus f is concave on $(-1, 1)$. [3]

Note that $f' > 0$ on $(-\infty, -1)$ and $(-1, 0)$. Hence f is strictly increasing on $(-\infty, -1)$ and $(-1, 0)$. Note that $f' < 0$ on $(1, \infty)$ and $(0, 1)$. Hence f is strictly decreasing on $(0, 1)$ and $(1, \infty)$. This shows that the point of local maximum is 0. [2]

4 (a) Compute $\lim_{t\to 0}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)$ and $\lim_{t\to 1}\frac{\sqrt{t+3}-2t}{\log t}$. [3+3=6]

Solution. $\star$ $\lim_{t\to 0}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{t-e^t+1}{t(e^t-1)}$ ($\frac{0}{0}$ form).

By L'Hospital's rule,

$$\lim_{t\to 0}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-e^t}{te^t+e^t-1}=\lim_{t\to 0}\frac{-e^t}{te^t+2e^t}=-\frac{1}{2}.$$

[3]

$\star$ $\lim_{t\to 1}\frac{\sqrt{t+3}-2t}{\log t}$ ($\frac{0}{0}$ form). By L'Hospital's rule,

$$\lim_{t\to 1}\frac{\sqrt{t+3}-2t}{\log t}=\lim_{t\to 1}\frac{\frac{1}{2\sqrt{t+3}}-2}{1/t}=-\frac{7}{4}.$$

[3]

(b) Show that $x-\frac{2}{3}x^3 \le \sin x\cos x \le x-\frac{2}{3}x^3+\frac{2}{15}x^5$ for all $x\in[0,\pi/4]$.

[6]

Solution. Let $f(x)=\sin x\cos x=\frac{\sin 2x}{2}$. Then $f^{(1)}(x)=\cos 2x$, $f^{(2)}(x)=-2\sin 2x=-4f(x)$. Thus

$$f^{(3)}(x)=-4f^{(1)}(x), f^{(4)}(x)=-4f^{(2)}(x), f^{(5)}(x)=-4f^{(3)}(x),$$

$$f^{(1)}(0)=1, f^{(2)}(0)=0, f^{(3)}(0)=-4, f^{(4)}(0)=0, f^{(5)}(0)=16.$$

[2]

By Taylor's Theorem, $f(x)=x-\frac{2}{3}x^3+\frac{2}{15}x^5\cos 2c$ for some $0<c<\pi/4$. [2]

Since $0\le\cos 2c\le 1$, we get

$$x-\frac{2}{3}x^3 \le f(x) \le x-\frac{2}{3}x^3+\frac{2}{15}x^5$$

for all $x\in[0,\pi/4]$. [2]

5 (a) Discuss the convergence of the following two series: [3+3=6]

$$\sum_{n=2}^{\infty} \frac{\log 2 + \cdots + \log n}{n} \text{ and } \sum_{n=1}^{\infty} n! \frac{1}{2^{n!}}.$$

Solution. $\star$ Note that $\frac{\log 2+\cdots+\log n}{n} \geq \frac{\log 2+\cdots+\log 2}{n} = \frac{n-1}{n}\log 2$ which does not converge to 0. Hence divergent.

Alternatively, this may also be done by applying limit comparison test to $a_n = \frac{\log 2+\cdots+\log n}{n}$ and $b_n = \frac{1}{n}$ (here $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$). [3]

$\star$ If $a_n = n!\frac{1}{2^{n!}}$ then $\frac{a_{n+1}}{a_n} = (n+1)\frac{1}{2^{n(n!)}}$. But $(n+1)\frac{1}{2^{n(n!)}} \to 0$ (by ratio test for sequences), and hence $\frac{a_{n+1}}{a_n}$ converges to 0. By ratio test for series, $\sum_{n=1}^{\infty} n!\frac{1}{2^{n!}}$ is convergent. [3]

(b) Let (a_n) be a decreasing sequence such that $a_n \to 0$. Show that [6]

$$a_1 - a_2 \;\leq\; \sum_{n=1}^{\infty}(-1)^{n+1}a_n \;\leq\; a_1.$$

Solution. By Leibniz Test, $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = l$ is convergent. Let S_n denote the nth partial sum of $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Thus $S_n \to l$. [2]

Since (a_n) is decreasing,

$$S_{2n} = a_1 + (a_3 - a_2) + \cdots + (a_{2n-1} - a_{2n-2}) - a_{2n} \leq a_1.$$

Letting $n \to \infty$, we get $l \leq a_1$. [2]

Since (a_n) is decreasing,

$$S_{2n+1} = a_1 - a_2 + (a_3 - a_4) + \cdots + (a_{2n} - a_{2n+1}) \geq a_1 - a_2.$$

Letting $n \to \infty$, we get $l \geq a_1 - a_2$. [2]

Thus we obtain $a_1 - a_2 \leq l \leq a_1$.

6 (a) Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{1+x}$ is integrable on $[0, 1]$. [6]

Solution. Consider the partition $P_n : \{0, 1/n, 2/n, \cdots, 1\}$ of $[0, 1]$.

Note that $U(P_n, f) = \sum_{i=1}^{n} \frac{M_i}{n}$, where

$$M_i = \sup_{x \in [(i-1)/n, i/n]} \frac{1}{1+x} = \frac{n}{n+i-1}.$$

Thus $U(P_n, f) = \sum_{i=1}^{n} \frac{1}{n+i-1}$. [2]

Note that $L(P_n, f) = \sum_{i=1}^{n} \frac{m_i}{n}$, where

$$m_i = \inf_{x \in [(i-1)/n, i/n]} \frac{1}{1+x} = \frac{n}{n+i}.$$

Thus $L(P_n, f) = \sum_{i=1}^{n} \frac{1}{n+i}$. [2]

It follows that $U(P_n, f) - L(P_n, f) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \to 0$. By Riemann's Criterion, f is integrable. [2]

(b) Show that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{k^2+n^2} = \log\sqrt{2}$. [6]

Solution. We apply the formula

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{f(k/n)}{n}$$

to the function $f(x) = \frac{x}{1+x^2}$. [2]

A simple calculation shows that $\sum_{k=1}^{\infty} \frac{f(k/n)}{n} = \sum_{k=1}^{n} \frac{k}{k^2+n^2}$. [2]

Also, $\int_0^1 f(x)dx = \int_0^1 \frac{d}{dx}\frac{\log(1+x^2)}{2}dx$. By Fundamental Theorem of Calculus, $\int_0^1 f(x)dx = \frac{\log 2}{2} = \log\sqrt{2}$. [2]

This gives $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{k^2+n^2} = \log\sqrt{2}$.