1 (a) Let \( x_1 = 1 \) and \( x_{n+1} = \frac{1}{2 + x_n} \) for \( n \in \mathbb{N} \). Show that \( (x_n) \) satisfies Cauchy criterion. Find the limit of \( (x_n) \). [6]

**Solution.** Note that

\[
|x_{n+2} - x_{n+1}| = \frac{1}{(2 + x_{n+1})(2 + x_n)} |x_n - x_{n+1}| < \frac{1}{4} |x_{n+1} - x_n|.
\]

Therefore \( (x_n) \) satisfies the contractive condition with \( \alpha = 1/4 \), and hence it satisfies the Cauchy criterion. [3]

If \( x_n \to l \) then \( l = \frac{1}{2 + l} \), which gives \( l^2 + 2l - 1 = 0 \). Hence \( (l + 1)^2 = 2 \), that is \( l + 1 = \pm \sqrt{2} \) or \( l = 1 \pm \sqrt{2} \). [2]

Since \( x_n > 0 \) for all \( n \), \( l = -1 + \sqrt{2} \). [1]

(b) Suppose the sequence \( (\cos nx) \) is convergent for some \( x \in \mathbb{R} \). Show that \( x = 2k\pi \) for some integer \( k \). [6]

**Solution.** Suppose that \( a_n := \cos nx \to l \). Since

\[
\cos(n + 1)x + \cos(n - 1)x = 2 \cos nx \cos x,
\]

by taking limit, we get \( 2l = 2l \cos x \). Thus \( l = 0 \) or \( \cos x = 1 \). [3]

Since \( \cos 2nx = 2 \cos^2 nx - 1 \), by taking limit, we get \( l = 2l^2 - 1 \). Hence \( l \neq 0 \). Hence \( \cos x = 1 \), that is, \( x = 2k\pi \) for some \( k \). [3]

**Note:** Give full marks if students use some other trigonometric identities to get the conclusion.
Let \( p(x) : \mathbb{R} \to \mathbb{R} \) be a polynomial of odd degree. Show that \( p(x) \) is an onto function. \([3+3=6]\)

**Solution.** Suppose \( p(x) = a_0 + a_1 x + \cdots + a_n x^{2n+1} \) and let \( \lambda \in \mathbb{R} \).

If \( a_n > 0 \) then \( p(x) \to \infty \) as \( x \to \infty \) and \( p(x) \to -\infty \) as \( x \to -\infty \).

If \( a_n < 0 \) then \( p(x) \to -\infty \) as \( x \to \infty \) and \( p(x) \to \infty \) as \( x \to -\infty \).

Thus there exists \( x_1, x_2 \) such that \( p(x_1) \leq \lambda \leq p(x_2) \). By IVP, there exists \( x_3 \) such that \( p(x_3) = \lambda \). \([3]\)

**Alternative Solution.** By IVP, every odd degree polynomial has a real root. Applying this to \( p(x) - \lambda \) for \( \lambda \in \mathbb{R} \), we get \( x_0 \) such that \( p(x_0) = \lambda \). \([3]\)

ii Show that there is no continuous function which maps \([0, 1]\) onto \((0, 1)\).

**Solution.** Suppose \( f \) is continuous such \( f([0, 1]) = (0, 1) \). In particular, \( 1 \notin f([0, 1]) \). Note that \( \sup_{x \in [0, 1]} f(x) = \sup(0, 1) = 1 \). Since \( f \) is continuous, \( 1 = \sup_{x \in [0, 1]} f(x) = f(x_0) \) for some \( x_0 \in [0, 1] \), which is not possible by our assumption. \([3]\)

**Alternative Solution.** If \( f \) is continuous then \( f([0, 1]) = [\inf f(x), \sup f(x)] \).

But \( \inf_{x \in [0, 1]} f(x) = 0 \) and \( \sup_{x \in [0, 1]} f(x) = 1 \). Thus we obtain \( f([0, 1]) = [0, 1] \), which is a contradiction. \([3]\)

(b) Let \( p(x) = a + bx + cx^2 \) be a quadratic polynomial. Find all values of \( a, b, c \in \mathbb{R} \) for which the function \( p(|x|) \) is differentiable at 0. \([6]\)

**Solution.** Note that \([3]\)

\[
\lim_{h \to 0} \frac{p(|h|) - p(0)}{h} = \lim_{h \to 0} \frac{a + b|h| + c|h|^2 - a}{h} = b \lim_{h \to 0} \frac{|h|}{h}.
\]

If \( b = 0 \) then clearly the derivative exists at 0. If \( b \neq 0 \) then by choosing paths \( h > 0 \) and \( h < 0 \), one can see that \( \lim_{h \to 0} \frac{p(|h|) - p(0)}{h} \) takes values \( b \) and \(-b\) respectively, so derivative does not exist. \([3]\)

Thus \( p(|x|) \) is differentiable at 0 for \( b = 0 \) and all values \( a, c \).
3 (a) A cylindrical box is to be made. The volume \( V \) of the box should be 250\( \pi \) cm. Find the height and radius of the box that minimizes the amount of material to be used. Recall: The volume \( V \) and surface area \( A \) are given by \( V = \pi r^2 h \) and \( A = 2\pi r^2 + 2\pi rh \) respectively. [6]

**Solution.** The volume is given by \( V = \pi r^2 h = 250\pi \). Thus \( h = \frac{V}{\pi r^2} = \frac{250}{r^2} \). The surface area is given by \( A = 2\pi r^2 + 2\pi rh \), where \( h \) is height and \( r \) is the radius of the can. This gives \( A = 2\pi r^2 + \frac{500}{r} \pi \). [3]

Note that \( \frac{dA}{dr} = 4\pi r - \frac{500}{r^2} \pi \) is negative if \( r < 5 \) and positive if \( r > 5 \). Thus \( A \) has minimum when \( r = 5 \). It follows that \( h = 10 \). [3]

(b) Find intervals of concavity/convexity and points of local minima/maxima of \( f(x) = \frac{x^2}{x^2 - 1} \) for \( x \neq 1 \). [6]

**Solution.** Note that \( f(x) = 1 + \frac{1}{x^2 - 1} \). Thus \( f'(x) = -\frac{2x}{(x^2 - 1)^2} \) and \( f''(x) = \frac{2(3x^2 + 1)}{(x^2 - 1)^3} \). [1]

Fact: If \( f''(x) > 0 \) (resp. \( f''(x) < 0 \)) for all \( x \in (a, b) \) then \( f \) is convex (resp. concave). Note that \( f'' > 0 \) on \((-\infty, -1) \) and \((1, \infty) \). Thus \( f \) is convex on \((-\infty, -1) \) and \((1, \infty) \). Note that \( f'' < 0 \) on \((-1, 1) \). Thus \( f \) is concave on \((-1, 1) \). [3]

Note that \( f' > 0 \) on \((-\infty, -1) \) and \((-1, 0) \). Hence \( f \) is strictly increasing on \((-\infty, -1) \) and \((-1, 0) \). Note that \( f' < 0 \) on \((0, 1) \) and \((1, \infty) \). Hence \( f \) is strictly decreasing on \((0, 1) \) and \((1, \infty) \). This shows that the point of local maximum is 0. [2]
4. (a) Compute \( \lim_{t \to 0} \left( \frac{1}{e^t-1} - \frac{1}{t} \right) \) and \( \lim_{t \to 1} \frac{\sqrt{t+3} - 2t}{\log t} \). [3+3=6]

**Solution.** \( * \lim_{t \to 0} \left( \frac{1}{e^t-1} - \frac{1}{t} \right) = \lim_{t \to 0} \frac{t-e^t}{t(e^t-1)} = \frac{0}{0} \) form. By L'Hospital's rule,

\[
\lim_{t \to 0} \left( \frac{1}{e^t-1} - \frac{1}{t} \right) = \lim_{t \to 0} \frac{-e^t}{te^t + e^t - 1} = \lim_{t \to 0} \frac{-e^t}{te^t + 2e^t} = -\frac{1}{2}.
\]

\[3\]

\( * \lim_{t \to 1} \frac{\sqrt{t+3} - 2t}{\log t} = \frac{0}{0} \) form. By L'Hospital's rule,

\[
\lim_{t \to 1} \frac{\sqrt{t+3} - 2t}{\log t} = \lim_{t \to 1} \frac{1}{2\sqrt{t+3} - 2/1} = -\frac{7}{4}.
\]

\[3\]

(b) Show that \( x - \frac{2}{3}x^3 \leq \sin x \cos x \leq x - \frac{2}{3}x^3 + \frac{2}{15}x^5 \) for all \( x \in [0, \pi/4] \). [6]

**Solution.** Let \( f(x) = \sin x \cos x = \frac{\sin 2x}{2} \). Then \( f^{(1)}(x) = \cos 2x, f^{(2)}(x) = -2\sin 2x = -4f(x) \). Thus

\[
f^{(3)}(x) = -4f^{(1)}(x), f^{(4)}(x) = -4f^{(2)}(x), f^{(5)}(x) = -4f^{(3)}(x),
\]

\[
f^{(1)}(0) = 1, f^{(2)}(0) = 0, f^{(3)}(0) = -4, f^{(4)}(0) = 0, f^{(5)}(0) = 16.
\]

By Taylor's Theorem, \( f(x) = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 \cos 2c \) for some \( 0 < c < \pi/4 \). [2]

Since \( 0 \leq \cos 2c \leq 1 \), we get

\[ x - \frac{2}{3}x^3 \leq f(x) \leq x - \frac{2}{3}x^3 + \frac{2}{15}x^5 \]

for all \( x \in [0, \pi/4] \). [2]
5 (a) Discuss the convergence of the following two series: \[3+3=6\]

\[
\sum_{n=2}^{\infty} \frac{\log 2 + \cdots + \log n}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n!}{2^n}.
\]

Solution. \* Note that \(\frac{\log 2 + \cdots + \log n}{n} \geq \frac{\log 2 + \cdots + \log 2}{n} = \frac{n-1}{n} \log 2\) which does not converge to 0. Hence divergent.

Alternatively, this may also be done by applying limit comparison test to \(a_n = \frac{\log 2 + \cdots + \log n}{n}\) and \(b_n = \frac{1}{n!}\) (here \(\lim_{n \to \infty} \frac{a_n}{b_n} = \infty\)).

\* If \(a_n = n! \frac{1}{2^n}\) then \(\frac{a_{n+1}}{a_n} = (n+1) \frac{1}{2^{n+1}}\). But \((n+1) \frac{1}{2^{n+1}} \to 0\) (by ratio test for sequences), and hence \(\frac{a_{n+1}}{a_n}\) converges to 0. By ratio test for series, \(\sum_{n=1}^{\infty} n! \frac{1}{2^n}\) is convergent.

(b) Let \((a_n)\) be a decreasing sequence such that \(a_n \to 0\). Show that \[6\]

\[
a_1 - a_2 \leq \sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq a_1.
\]

Solution. By Leibniz Test, \(\sum_{n=1}^{\infty} (-1)^{n+1} a_n \) is convergent. Let \(S_n\) denote the \(n\)th partial sum of \(\sum_{n=1}^{\infty} (-1)^{n+1} a_n\). Thus \(S_n \to l\). \[2\]

Since \((a_n)\) is decreasing,

\[
S_{2n} = a_1 + (a_3 - a_2) + \cdots + (a_{2n-1} - a_{2n-2}) - a_{2n} \leq a_1.
\]

Letting \(n \to \infty\), we get \(l \leq a_1\). \[2\]

Since \((a_n)\) is decreasing,

\[
S_{2n+1} = a_1 - a_2 + (a_3 - a_4) + \cdots + (a_{2n} - a_{2n+1}) \geq a_1 - a_2.
\]

Letting \(n \to \infty\), we get \(l \geq a_1 - a_2\). \[2\]

Thus we obtain \(a_1 - a_2 \leq l \leq a_1\).
6 (a) Using Riemann’s criterion for the integrability, show that \( f(x) = \frac{1}{1+x} \) is integrable on \([0, 1]\).

**Solution.** Consider the partition \( P_n : \{0, 1/n, 2/n, \cdots, 1\} \) of \([0, 1]\).

Note that \( U(P_n, f) = \sum_{i=1}^{n} M_i \), where
\[
M_i = \sup_{x \in [(i-1)/n, i/n]} \frac{1}{1 + x} = \frac{n}{n + i - 1}.
\]
Thus \( U(P_n, f) = \sum_{i=1}^{n} \frac{1}{n + i - 1} \).

Note that \( L(P_n, f) = \sum_{i=1}^{n} m_i \), where
\[
m_i = \inf_{x \in [(i-1)/n, i/n]} \frac{1}{1 + x} = \frac{n}{n + i}.
\]
Thus \( L(P_n, f) = \sum_{i=1}^{n} \frac{1}{n + i} \).

It follows that \( U(P_n, f) - L(P_n, f) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \to 0 \). By Riemann’s Criterion, \( f \) is integrable.

(b) Show that \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{k^2 + n^2} = \log \sqrt{2} \).

**Solution.** We apply the formula
\[
\int_0^1 f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{f(k/n)}{n}
\]
to the function \( f(x) = \frac{x}{1+x^2} \).

A simple calculation shows that \( \sum_{k=1}^{\infty} \frac{f(k/n)}{n} = \sum_{k=1}^{n} \frac{k}{k^2 + n^2} \).

Also, \( \int_0^1 f(x) \, dx = \int_0^1 \frac{d}{dx} \frac{\log(1+x^2)}{2} \, dx \). By Fundamental Theorem of Calculus, \( \int_0^1 f(x) \, dx = \frac{\log 2}{2} = \log \sqrt{2} \).

This gives \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{k^2 + n^2} = \log \sqrt{2} \).