MID SEMESTER EXAMINATION

MTH 101A, 18TH OCTOBER 2020

Time: 2 hrs (13:00-15:00hrs) **Full Marks: 60**

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(1) (a) Let $x_1 = \sqrt{2}$ and x_n be defined recursively as $x_n = \sqrt{2 + x_{n-1}}$. Show that the sequence $\{x_n\}$ converges as $n \to \infty$. Find $\lim_{n\to\infty} x_n$. [4]

Soln: $\{x_n\}$ is bounded (by induction): $0 \le x_n \le 2$ and [1]
$\{x_n\}$ is increasing (by induction) [2]
If the limit is l then $(l-2)(l+1) = 0$. $l \neq -1$ as $x_n \geq 0$ thus $l = 2$ [1]

(b) Let $f(x) = \begin{cases} x(x-1) & if\ x \in \mathbb{Q} \\ 0 & if\ x \notin \mathbb{Q}. \end{cases}$ Determine all the points in which f is discontinuous. Justify your answer. [4]

Soln: Let $x \in \mathbb{Q}$ but $x \neq 0, 1$. Consider $\{x_n\} \subset \mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x$.
Then $0 = f(x_n) \not\to x(x-1)$. [1]
Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Consider $\{x_n\} \subset \mathbb{Q}$ such that $x_n \to x$.
Then $f(x_n) = x_n(x_n - 1) \to x(x-1) \neq 0$. [1]
For $x = 0$ and $x_n \to x$ then $|f(x_n)| \leq 2|x_n| \to 0$ [1]
For $x = 1$ and $x_n \to x$ then $|f(x_n)| \leq 2|x_n - 1| \to 0$ [1]

(c) Let $f : [0, 20] \to \mathbb{R}$ be a continuous function and $f(0) = f(10)$. Show that there exist real numbers $x_1, x_2 \in [0, 20]$ such that $x_1 - x_2 = 5$ and $f(x_1) = f(x_2)$. [4]

Soln: Consider $g(x) = f(x+5) - f(x)$. Then g is continuous. [1]
Note that $g(0) = f(5) - f(0)$ and $g(5) = f(10) - f(5) = f(0) - f(5)$.
So $\exists\ x_1 \in (0, 5)$ such that $g(x_1) = 0$ (by IVP). [2]
Thus, $f(x_1 + 5) = f(x_1)$. Take $x_2 = x_1 + 5$. [1]

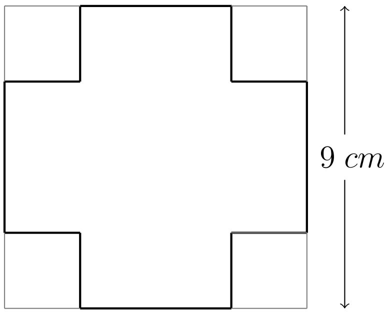
(2) (a) Using Cauchy mean value theorem prove that $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$ for $x > 0$. [4]

Soln: Ist ineq: $\exists\ \xi \in (0, x)$ such that $\frac{x - \frac{x^2}{2}}{\log(1+x)} = \frac{1-\xi}{\frac{1}{1+\xi}} = 1 - \xi^2 < 1$ [2]
2nd Ineq: Consider $g(x) = x - \frac{x^2}{2(1+x)}$. Then $g'(x) = \frac{2+2x+x^2}{2(1+x)^2}$.
Thus $\frac{\log(1+x)}{x - \frac{x^2}{2(1+x)}} = \frac{2(1+\xi)}{2+2\xi+\xi^2} < 1$. [2]

(b) Let $f(x) = (x^2 - 3)^2 P(x)$ where P is a polynomial of degree 4. Show that there exist at least two distinct values of x for which $f''(x) = 0$. [4]

Soln: $f(\pm\sqrt{3}) = 0$. So f' has a zero in $(-\sqrt{3}, \sqrt{3})$. [1]
$f'(x) = 2(x^2-3).2x.P(x) + (x^2-3)^2 P'(x) = (x^2-3)(4xP(x) + (x^2-3)^2 P'(x))$. [1]
Thus f' has also zero in $\pm\sqrt{3}$ i.e. f' has three zeros. [1]
Thus f'' has at least two zeros. [1]

(c) An open-top box is made from a square sheet of cardboard by cutting a small square of cardboard from each corners and bending up sides. Each side of cardboard is 9 *cm*. What is the possible maximum volume of the box? [4]



Soln: Let x be the length of the corner square. Then $length = width = 9 - 2x$ and $height = x$. So Volume $V(x) = (9-2x)^2 x$. [1]
$V'(x) = (9-2x)(9-6x)$
Critical points $x = \frac{9}{2}$ and $x = \frac{9}{6}$. [1]
$x = \frac{9}{6}$ maximum [1]
$V(\frac{9}{6}) = (9-3)^2 \times \frac{3}{2} = 54$ [1]

(3) (a) Consider the function $f(x) = \frac{x^3}{9-x^2}$. [6]

(i) Find slant asymptotes if any.
(ii) Determine the interval in which f is increasing or decreasing.
(iii) Find local maximum(s) or minimum(s).
(iv) Find the interval of concavity and concavity.
(v) Sketch the curve.

Soln: Vertical asymptotes: $x = \pm 3$.
As $\frac{x^3}{9-x^2} + x = \frac{9x}{9-x^2} \to 0$ when $x \to \pm\infty$, $y = -x$ is a slant asymptote. [1]
$f'(x) = \frac{x^2(27-x^2)}{(9-x^2)^2}$. So f is increasing on $(-3\sqrt{3}, -3) \cup (-3, 3) \cup (3, 3\sqrt{3})$ and decreasing $|x| > 3\sqrt{3}$. [1]
Critical points $\pm 3\sqrt{3}$. Local max at $3\sqrt{3}$ and local minimum at $-3\sqrt{3}$. [1]
$f''(x) = \frac{18x(x^2+27)}{(9-x^2)^3}$. Convex on $(-\infty, -3) \cup (0, 3)$ and concave on $(-3, 0) \cup (3, \infty)$. [1]
Sketch [2]

(b) Show that the function $f(x) = \sin x$, $x \in [0, \frac{\pi}{4}]$ is approximated by a polynomial $p(x) = x - \frac{x^3}{6}$ with an error term less than $\frac{1}{200}$. [3]

Soln: By Taylor's theorem $f(x) = \sin x = x - \frac{x^3}{6} + \frac{\cos c}{5!} x^5$ [1]
Error term $E(x) = f(x) - P(x) = \frac{\cos c}{5!} x^5$ [1]
$|E(x)| \leq \frac{\pi^5}{120 \times 4^5} \leq \frac{1}{200}$. [1]

(c) Use Newton-Raphson to find first three approximation for $2^{\frac{1}{4}}$ by considering first approximation to be 1. [3]

Soln: Let $f(x) = x^4 - 2$ then $f'x) = 4x^3$. [1]
$x_1 = 1$. So $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{-1}{4} = \frac{5}{4}$. [1]
$x_3 = \frac{5}{4} - \frac{f(\frac{5}{4})}{f'(\frac{5}{4})} = \frac{5}{4} - \frac{625-512}{16 \times 125}$. [1]

(4) (a) Let $a_n > 0$ for all n and the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges. Show that the series $\sum_{n=1}^{\infty} a_n$ converges. [4]

Soln: $\exists N_0$ such that $\frac{a_n}{1+a_n} < \frac{1}{2} \ \forall \ n > N_0$. [1]
Therefore $a_n \leq 1$ and $\frac{1}{1+a_n} \geq \frac{1}{2} \ \forall \ n > N_0$ [1]
$\frac{a_n}{1+a_n} \geq \frac{1}{2} a_n$. [2]

(b) Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ converges. [4]

Soln: By Cauchy condensation test for non-negative decreasing a_n,
$\sum a_n$ and $\sum 2^k a_{2^k}$ converges diverges simultaneously. [1]
Realising positive and decreasing [1]
$\sum 2^k \frac{1}{2^k k^2 (\log 2)^2} = \frac{1}{(\log 2)^2} \sum \frac{1}{k^2}$ is a convergent series. [2]

(c) Determine all values of x for which the series $\sum_{n=1}^{\infty} \frac{n^{xn}}{4^{2n}}$ converges. [4]

Soln: By Root test $a_n^{\frac{1}{n}} = \frac{n^x}{4^2}$. [1]
Thus if $x \leq 0$ it goes to zero so converges. [2]
If $x > 0$ it goes to ∞ so diverges. [1]

(5) (a) Let $f : [0,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{1}{2^n} & if\ x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n}], \qquad n = 0, 1, 2, \ldots \\ 0 & if\ x = 0. \end{cases}$$

Show that $\int_0^1 f(x)dx = \frac{2}{3}$. [4]

Soln:

$$\int_0^1 f(x)dx = \sum_{n=0}^{\infty} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} f(x)dx \quad [1]$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} dx \quad [1]$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\frac{1}{2^n} - \frac{1}{2^{n+1}} \right) \quad [1]$$

$$= \frac{1}{2} \times \frac{4}{3} \quad [1]$$

(b) Let f be continuous function on $[0,1]$ and $\int_0^1 f(x)dx = 0$. Show that there exists at least one $c \in [0,1]$ such that $f(c) = 0$. [4]

Soln: Define $F(x) = \int_0^x f(t)dt$ [2]
$F(0) = F(1) = 0$ [1]
$\exists c \in (0,1)$ such that $F'(c) = f(c) = 0$. [1]

(c) Let f be a function such that for every $n \in \mathbb{N}$ the integral $\int_1^n f((x)dx = I_n$ exists. If $\lim_{n\to\infty} I_n$ exists then is it necessarily true that $\int_1^{\infty} f(x)dx$ exists as an improper integral?
If answer is yes then prove it. If answer is no then give a counter example. [4]
Soln: Consider $f(x) = \sin 2\pi x$ [1]
$\int_1^n f(x)dx = 0$ hence $\lim_{n\to\infty} \int_1^n f(x)dx = 0$ [1]

However,

$$\begin{aligned}\int_1^{n+\frac{1}{2}} f(x)dx &= \int_n^{n+\frac{1}{2}} \sin 2\pi x dx \\ &= -\frac{1}{2\pi}\left[\cos 2\pi(n+\frac{1}{2}) - \cos 2\pi n\right] \\ &= \frac{1}{2\pi} \not\to 0. \qquad [2]\end{aligned}$$