1. (a) Consider a sequence of positive real numbers $(x_n)$ such that $1 \leq x_1 \leq x_2 \leq 3$ and $x_{n+1}^2 = x_n x_{n-1}$ for all $n \geq 2$. Show that $(x_n)$ is a Cauchy sequence. [5]

**Solution:** Observe that $x_{n+1}^2 = x_n x_{n-1} = x_n (x_{n-1} - x_n)$. Therefore $|x_{n+1} - x_n| = \frac{x_{n} - x_{n-1}}{x_n} |x_{n-1} - x_n|$. [1]

Since $1 \leq x_n \leq 3$ for all $n$ and $\frac{x_{n+1}}{x_{n}} \geq \frac{1}{3}$, $\frac{x_{n+1} + x_n}{x_{n}} \geq \frac{4}{3}$. [2]

Therefore, $|x_{n+1} - x_n| \leq \frac{3}{4} |x_{n-1} - x_n|$. [1]

(b) Let $x_n = 2 + (-1)^n$ for $n \in \mathbb{N}$. Show that $\lim_{n \to \infty} (x_1 x_2 \ldots x_n)^{\frac{1}{n}} = \sqrt{3}$. [5]

**Solution:** Let $y_n = (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$. Then $y_{2n-1} = (3^{n-1})^{\frac{1}{3n-1}}$ for $n \geq 1$ [2] and $y_{2n} = (3^n)^{\frac{1}{3n}}$ for $n \geq 1$. [2]

Note that $y_{2n} \to \sqrt{3}$ and $y_{2n-1} \to \sqrt{3}$ and therefore, $y_n \to \sqrt{3}$. [1]

(c) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that $f'(x) < 1$ for all $x \in [0,1]$ and $f(0) = 1$. Show that $f(1) < 2$. [2]

**Solution:** By MVT, $f(1) - f(0) = f'(c)$ for some $c \in [0,1]$. [2]

2. (a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) = f(\sqrt{x})$ for all $x \in [0,1]$. Show that $f$ is constant. [4]

**Solution:** For $x \in (0,1]$, note that $f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{3}}) = f(x^{\frac{1}{4}})$. [2]

Since $x^{\frac{1}{4}} \to 1$, $f(x^{\frac{1}{4}}) \to f(1)$ and hence $f(x) = f(1)$. [1]

By continuity of $f$, $\lim_{x \to 0} f(x) = f(0) = f(1)$. [1]

(b) Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Determine the set of points in which $f$ is continuous. [5]

**Solution:** At $x_0 = 0$: Note that $|f(x) - f(0)| \leq |x - 0|$ for all $x \in \mathbb{R}$. As $f(x_n) \to f(0)$ whenever $x_n \to 0$, $f$ is continuous at 0. [2]

Suppose $x_0$ is non-zero rational.

Then there exists a sequence of irrationals $(x_n)$ such that $x_n \to x_0$. [1]

Since $f(x_n) = x_n \to x_0 \neq f(x_0)$, $f$ is not continuous at $x_0$. [1]

Suppose $x_0$ is irrational.

Then there exists a sequence of rationals $(x_n)$ such that $x_n \to x_0$.

Since $f(x_n) = 0 \to 0 \neq f(x_0)$, $f$ is not continuous at $x_0$. [1]

(c) Let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = \frac{3-4x}{1+4x}$. Show that there exists $x_0 \in (0,\infty)$ such that $f(x_0) = \frac{5}{3}$. [3]

**Solution:** This question will not be evaluated. However, 3 marks will be added to all the students after the evaluation.

3. (a) Let $P(x)$ be a polynomial which has at least two distinct real roots. Show that $P'(x) + 10P(x)$ has a real root. [5]
Solution: Suppose \( P(x_0) = P(y_0) = 0 \) for some \( x_0, y_0 \in \mathbb{R} \) and \( x_0 \neq y_0 \).

Let \( g(x) = P(x)e^{10x} \). \[1\]

Since \( g(x_0) = g(y_0) = 0, \exists z_0 \in \mathbb{R} \) such that \( g'(z_0) = 0 \). \[1\]

Note that \( g'(z_0) = e^{10z_0}[P'(z_0) + 10P(z_0)] \). \[1\]

Therefore, \( P'(z_0) + 10P(z_0) = 0 \). \[1\]

(b) Determine the values of \( x \) for which the power series \( \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n \log n)^2} \) converges.

Solution: If for a fixed \( x \), \( a_n = \frac{(-1)^n x^n}{n(n \log n)^2} \), then \( \frac{a_{n+1}}{a_n} \to |x| \). \[2\]

Hence the power series converges for \( |x| < 1 \). \[1\]

When \( x = 1 \), the series converges by Leibniz test. \[1\]

When \( x = -1 \), the series is \( \sum_{n=2}^{\infty} \frac{1}{n(n \log n)^2} \) which diverges. \[1\]

(c) Let \( P(x) = a + bx + cx^2 \) where \( a, b \) and \( c \) are non-zero real numbers. Define \( f(x) = P(x|x|) \) for all \( x \in \mathbb{R} \). Show that \( f \) is differentiable at 0.

Solution: \( \lim_{t \to 0} \frac{f(t) - f(0)}{t} = \lim_{t \to 0} \frac{at + bt^2}{t} = 0. \) \[2\]

4. (a) Using Cauchy mean value theorem, show that \( e^{-x^3} \geq 1 - x^3 \) for \( x > 0 \).

Solution: Let \( x > 0 \).

By CMVT, there exists \( c > 0 \) such that \( \frac{e^{-x^3} - e^0}{1-x^3-1} = \frac{-3x^2e^{-c^3}}{-3x^2} \). \[2\]

Since \( e^{-c^3} < 1 \), \( e^{-x^3} \geq 1 - x^3 \). \[2\]

(b) Starting with the initial value \( x_1 = -1 \), using Newton-Raphson method, find the third approximation \( x_3 \) to a real root of the equation \( x^3 - 4x^2 - x + 2 = 0 \) up to three decimal places.

Solution: Let \( f(x) = x^3 - 4x^2 - x + 2 \). Then \( f'(x) = 3x^2 - 8x - 1 \).

By Newton-Raphson method, \( x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \) \[1\]

Hence \( x_2 = (-1) - \frac{(-1)^3 - 4 + 1 + 2}{3 + 8 - 1} = -0.8 \) \[1\]

\( x_3 = (-0.8) - \frac{(-0.8)^3 - 4(0.8)^2 + 0.8 + 2}{3(0.8)^2 + 8(0.8) - 1} \approx -0.763 \) \[1\]

(c) Let \( f : [-1, 1] \to \mathbb{R} \) be a twice differentiable function such that \( f'(0) = 0 \) and \( f(1) = f(-1) \). Show that there exist \( c_1 \in [-1, 0] \) and \( c_2 \in [0, 1] \) such that \( f''(c_1) = f''(c_2) \).

Solution: By EMVT, \( f(1) = f(0) + f'(0) + \frac{f''(c_1)}{2} \) for some \( c_1 \in [0, 1] \) \[2\]

and \( f(-1) = f(0) - f'(0) + \frac{f''(c_2)}{2} \) for some \( c_2 \in [-1, 0] \) \[2\]

Hence \( \frac{f''(c_1)}{2} = f(1) - f(0) = f(-1) - f(0) = \frac{f''(c_2)}{2} \) \[1\]

5. (a) Discuss the convergence/divergence of the following series:

\[ \sum_{n=1}^{\infty} \left( \sin \frac{1}{n} - \sin \frac{1}{n+2} \right) \] \[4\]

Solution: The partial sum
\[ S_n = (\sin 1 - \sin \frac{1}{3}) + (\sin \frac{1}{2} - \sin \frac{1}{4}) + \cdots + (\sin \frac{1}{n} - \sin \frac{1}{n+2}). \]  
[2]

Note that \( S_n = \sin 1 + \sin \frac{1}{2} - \sin \frac{1}{n+1} - \sin \frac{1}{n+2} \).  
[1]

Since \((S_n)\) converges, the series converges.  
[1]

\( \text{OR} \)

Note that \(| \sin \frac{1}{n} - \sin \frac{1}{n+2} | \leq | \frac{1}{n} - \frac{1}{n+2} | \).  
[2]

Since \( \sum_{n=1}^{\infty} \frac{2}{n(n+2)} \) converges, the series converges absolutely.  
[1]

Since Abs. Conv. \( \Rightarrow \) Conv., \( \sum_{n=1}^{\infty} (\sin \frac{1}{n} - \sin \frac{1}{n+2}) \) converges.  
[1]

(b) Let \( a_n > 0 \) for all \( n \in \mathbb{N} \) and \( \sum a_n \) converge. Show that \( \sum_{n=1}^{\infty} a_n (a_1 + a_2 + \cdots + a_n) \) converges.  
[5]

**Solution:** Let \((S_n)\) denote the partial sum of \( \sum_{n=1}^{\infty} a_n \) and let \( S_n \to S \).  
[1]

Since \((S_n)\) is increasing \( S_n \leq S \) for all \( n \).  
[1]

Hence \( a_n (a_1 + a_2 + \cdots + a_n) \leq a_n S \).  
[2]

Since \( \sum_{n=1}^{\infty} (a_n S) \) converges, \( \sum_{n=1}^{\infty} a_n (a_1 + a_2 + \cdots + a_n) \) converges,  
[1]

by comparison test.

(c) Show that \( x^{15} + 100x^5 - 5e^{-x} = 0 \) has exactly one real solution.  
[3]

**Solution:** Let \( f(x) = x^{15} + 100x^5 - 5e^{-x} \).

Since \( f(0) < 0 \) and \( f(1) > 0 \), by IVP, \( f(x) = 0 \) has a real solution.  
[1]

Note that \( f'(x) > 0 \) for all \( x \).

Hence, \( f(x) = 0 \) has exactly one real solution as \( f \) is strictly increasing.  
[2]

\( \text{OR} \) If \( f(x) = 0 \) has two real solutions, then by Rolle’s theorem \( f'(x_0) = 0 \) which is not true.  
[2]

6. (a) Consider the function \( f(x) = \frac{2x^3 - x^2 + 9}{2(9-x^2)} \).

(i) Find the vertical and slant asymptotes if any.

(ii) Determine the intervals in which \( f \) is increasing/decreasing.

(iii) Find the points of local maximum and local minimum.

(iv) Find the intervals of convexity/concavity and point of inflection.

(v) Sketch the graph.

**Solution:** Vertical asymptotes: \( x = \pm 3 \).  
[1]

Observe that \( \frac{2x^3 - x^2 + 9}{2(9-x^2)} = (-x + \frac{1}{2}) + \frac{9x}{9-x^2} \).  
[1]

Note that \( y = -x + \frac{1}{2} \) is a slant asymptote.

\( f'(x) = \frac{x^2(27-x^2)}{(9-x^2)^2} \). So \( f \) is increasing on \((-3\sqrt{3}, -3) \cup (-3, 3) \cup (3, 3\sqrt{3})\) and decreasing \(|x| > 3\sqrt{3} \).  
[1]

Critical points \( \pm 3\sqrt{3} \). Local max at \( 3\sqrt{3} \) and local minimum at \(-3\sqrt{3} \).  
[1]

\( f''(x) = \frac{18x(x^2+27)}{(9-x^2)^3} \). Convex on \((-\infty, -3) \cup (0, 3)\) and concave on \((-3, 0) \cup (3, \infty)\).  
[1]

0 is the point of inflection.  
[1]
(b) Let \( f : [0, 1] \rightarrow \mathbb{R} \). Suppose for each \( x \in [0, 1] \) there exists \( y \in [0, 1] \) such that \( |f(y)| \leq |f(x)| \). Show that there is a sequence \((x_n)\) in \([0, 1]\) such that \(|f(x_n)|\) converges.

**Solution:** Find \((x_n)\) such that \( \cdots \leq |f(x_n)| \leq |f(x_2)| \leq |f(x_1)| \). Since \((|f(x_n)|)\) is decreasing and bounded below by 0, it converges.