MTH101AA (2020), Tentative Marking Scheme - Mid sem. exam

1. (a) Consider a sequence of positive real numbers (x_n) such that $1 \leq x_1 \leq x_2 \leq 3$ and $x_{n+1}^2 = x_n x_{n-1}$ for all $n \geq 2$. Show that (x_n) is a Cauchy sequence. [5]

   Solution: Observe that $x_{n+1}^2 - x_n^2 = x_n x_{n-1} - x_n^2 = x_n(x_{n-1} - x_n)$. [1]
   Therefore $|x_{n+1} - x_n| = |\frac{x_n}{x_{n+1}+x_n}||x_{n-1} - x_n|$. [1]
   Since $1 \leq x_n \leq 3$ for all n and $\frac{x_{n+1}}{x_n} \geq \frac{1}{3}$, $\frac{x_{n+1}+x_n}{x_n} \geq \frac{4}{3}$. [2]
   Therefore, $|x_{n+1} - x_n| \leq \frac{3}{4}|x_{n-1} - x_n|$. [1]

   (b) Let $x_n = 2 + (-1)^n$ for $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} (x_1 x_2 \dots x_n)^{\frac{1}{n}} = \sqrt{3}$. [5]

   Solution: Let $y_n = (x_1 x_2 \dots x_n)^{\frac{1}{n}}$. Then $y_{2n-1} = (3^{n-1})^{\frac{1}{2n-1}}$ for $n \geq 1$ [2]
   and $y_{2n} = (3^n)^{\frac{1}{2n}}$ for $n \geq 1$. [2]
   Note that $y_{2n} \to \sqrt{3}$ and $y_{2n-1} \to \sqrt{3}$ and therefore, $y_n \to \sqrt{3}$. [1]

   (c) Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that $f'(x) < 1$ for all $x \in [0, 1]$ and $f(0) = 1$. Show that $f(1) < 2$. [2]

   Solution: By MVT, $f(1) - f(0) = f'(c)$ for some $c \in [0, 1]$. [2]

2. (a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) = f(\sqrt{x})$ for all $x \in [0, 1]$. Show that f is constant. [4]

   Solution: For $x \in (0, 1]$, note that $f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{2^2}}) = f(x^{\frac{1}{2^n}})$. [2]
   Since $x^{\frac{1}{2^n}} \to 1$, $f(x^{\frac{1}{2^n}}) \to f(1)$ and hence $f(x) = f(1)$. [1]
   By continuity of f, $\lim_{x\to 0} f(x) = f(0) = f(1)$. [1]

   (b) Let $f(x) = \begin{cases} 0 & if\ x \in \mathbb{Q} \\ x & if\ x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Determine the set of points in which f is continuous. [5]

   Solution: At $x_0 = 0$: Note that $|f(x) - f(0)| \leq |x - 0|$ for all $x \in \mathbb{R}$.
   As $f(x_n) \to f(0)$ whenever $x_n \to 0$, f is continuous at 0. [2]
   Suppose x_0 is non-zero rational.
   Then there exists a sequence of irrationals (x_n) such that $x_n \to x_0$. [1]
   Since $f(x_n) = x_n \to x_0 \neq f(x_0)$, f is not continuous at x_0. [1]
   Suppose x_0 is irrational.
   Then there exists a sequence of rationals (x_n) such that $x_n \to x_0$.
   Since $f(x_n) = 0 \to 0 \neq f(x_0)$, f is not continuous at x_0. [1]

   (c) Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{3-4^{\frac{1}{x}}}{1+4^{\frac{1}{x}}}$. Show that there exists $x_0 \in (0, \infty)$ such that $f(x_0) = \frac{5}{2}$. [3]

   Solution: This question will not be evaluated. However, 3 marks will be added to all the students after the evaluation.

3. (a) Let $P(x)$ be a polynomial which has at least two distinct real roots. Show that $P'(x) + 10P(x)$ has a real root. [5]

Solution: Suppose $P(x_0) = P(y_0) = 0$ for some $x_0, y_0 \in \mathbb{R}$ and $x_0 \neq y_0$.
Let $g(x) = P(x)e^{10x}$. [2]
Since $g(x_0) = g(y_0) = 0$, $\exists\ z_0 \in \mathbb{R}$ such that $g'(z_0) = 0$. [1]
Note that $g'(z_0) = e^{10z_0}[P'(z_0) + 10P(z_0)]$. [1]
Therefore, $P'(z_0) + 10P(z_0) = 0$. [1]

(b) Determine the values of x for which the power series $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(\log n)^{\frac{1}{2}}}$ converges. [5]

Solution: If for a fixed x, $a_n = \frac{(-1)^n x^n}{n(\log n)^{\frac{1}{2}}}$, then $\left|\frac{a_{n+1}}{a_n}\right| \to |x|$. [2]
Hence the power series converges for $|x| < 1$. [1]
When $x = 1$, the series converges by Leibniz test. [1]
When $x = -1$, the series is $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\frac{1}{2}}}$ which diverges. [1]

(c) Let $P(x) = a + bx + cx^2$ where a, b and c are non-zero real numbers. Define $f(x) = P(x|x|)$ for all $x \in \mathbb{R}$. Show that f is differentiable at 0. [2]

Solution: $lim_{t\to 0}\frac{f(t)-f(0)}{t} = \lim_{t\to 0} \frac{a+bt|t|+ct^4-a}{t} = 0$. [2]

4. (a) Using Cauchy mean value theorem, show that $e^{-x^3} \geq 1 - x^3$ for $x > 0$. [3]

Solution: Let $x > 0$.
By CMVT, there exists $c > 0$ such that $\frac{e^{-x^3}-e^0}{1-x^3-1} = \frac{-3c^2e^{-c^3}}{-3c^2}$. [2]
Since $e^{-c^3} < 1$, $e^{-x^3} \geq 1 - x^3$. [1]

(b) Starting with the initial value $x_1 = -1$, using Newton-Raphson method, find the third approximation x_3 to a real root of the equation $x^3 - 4x^2 - x + 2 = 0$ up to three decimal places. [4]

Solution: Let $f(x) = x^3 - 4x^2 - x + 2$. Then $f'(x) = 3x^2 - 8x - 1$.
By Newton-Raphson method, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ [1]
Hence $x_2 = (-1) - \frac{(-1)^3-4+1+2}{3+8-1}$ [1]
$= -0.8$
$x_3 = (-0.8) - \frac{(-0.8)^3-4(0.8)^2+0.8+2}{3(0.8)^2+8(0.8)-1}$ [1]
≈ -0.763 [1]

(c) Let $f : [-1, 1] \to \mathbb{R}$ be a twice differentiable function such that $f'(0) = 0$ and $f(1) = f(-1)$. Show that there exist $c_1 \in [-1, 0]$ and $c_2 \in [0, 1]$ such that $f''(c_1) = f''(c_2)$. [5]

Solution: By EMVT, $f(1) = f(0) + f'(0) + \frac{f''(c_1)}{2}$ for some $c_1 \in [0, 1]$ [2]
and $f(-1) = f(0) - f'(0) + \frac{f''(c_2)}{2}$ for some $c_2 \in [-1, 0]$ [2]
Hence $\frac{f''(c_1)}{2} = f(1) - f(0) = f(-1) - f(0) = \frac{f''(c_2)}{2}$ [1]

5. (a) Discuss the convergence/divergence of the following series:
$\sum_{n=1}^{\infty} \left(\sin \frac{1}{n} - \sin \frac{1}{n+2}\right)$ [4]

Solution: The partial sum

$S_n = (sin 1 - sin\frac{1}{3}) + (sin\frac{1}{2} - sin\frac{1}{4}) + \cdots + (sin\frac{1}{n} - sin\frac{1}{n+2})$. [2]
Note that $S_n = sin 1 + sin\frac{1}{2} - sin\frac{1}{n+1} - sin\frac{1}{n+2}$ [1]
Since (S_n) converges, the series converges. [1]
OR
Note that $|\sin\frac{1}{n} - \sin\frac{1}{n+2}| \le |\frac{1}{n} - \frac{1}{n+2}|$. [2]
Since $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ converges, the series converges absolutely . [1]
Since Abs. Conv. $\Rightarrow$ Conv., $\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+2}\right)$ converges. [1]

(b) Let $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converge. Show that $\sum_{n=1}^{\infty} a_n(a_1+a_2+\cdots+a_n)$ converges. [5]

Solution: Let (S_n) denote the partial sum of $\sum_{n=1}^{\infty} a_n$ and let $S_n \to S$. [1]
Since (S_n) is increasing $S_n \le S$ for all n. [1]
Hence $a_n(a_1 + a_2 + \cdots + a_n) \le a_n S$. [2]
Since $\sum_{n=1}^{\infty} (a_n S)$ converges, $\sum_{n=1}^{\infty} a_n(a_1 + a_2 + \cdots + a_n)$ converges, [1]
by comparison test.

(c) Show that $x^{15} + 100x^5 - 5e^{-x} = 0$ has exactly one real solution. [3]

Solution: Let $f(x) = x^{15} + 100x^5 - 5e^{-x}$.
Since$f(0) < 0$ and $f(1) > 0$, by IVP, $f(x) = 0$ has a real solution. [1]
Note that $f'(x) > 0$ for all x.
Hence, $f(x) = 0$ has exactly one real solution as f is strictly increasing. [2]
OR If $f(x) = 0$ has two real solutions, then by Rolle's theorem $f'(x_0) = 0$ which is not true. [2]

6. (a) Consider the function $f(x) = \frac{2x^3-x^2+9}{2(9-x^2)}$.
(i) Find the vertical and slant asymptotes if any.
(ii) Determine the intervals in which f is increasing/decreasing.
(iii) Find the points of local maximum and local minimum.
(iv) Find the intervals of convexity/concavity and point of inflection.
(v) Sketch the graph. [8]

Solution: Vertical asymptotes: $x = \pm 3$. [1]
Observe that $\frac{2x^3-x^2+9}{2(9-x^2)} = (-x + \frac{1}{2}) + \frac{9x}{9-x^2}$ [1]
Note that $y = -x + \frac{1}{2}$ is a slant asymptote. [1]
$f'(x) = \frac{x^2(27-x^2)}{(9-x^2)^2}$. So f is increasing on $(-3\sqrt{3}, -3) \cup (-3, 3) \cup (3, 3\sqrt{3})$ and decreasing $|x| > 3\sqrt{3}$. [1]
Critical points $\pm 3\sqrt{3}$. Local max at $3\sqrt{3}$ and local minimum at $-3\sqrt{3}$. [1]
$f''(x) = \frac{18x(x^2+27)}{(9-x^2)^3}$. Convex on $(-\infty, -3) \cup (0, 3)$ and concave on $(-3, 0) \cup (3, \infty)$. [1]
0 is the point of inflection [1]

Sketch [1]

(b) Let $f : [0,1] \to \mathbb{R}$. Suppose for each $x \in [0,1]$ there exists $y \in [0,1]$ such that $|f(y)| \leq |f(x)|$. Show that there is a sequence (x_n) in $[0,1]$ such that $(|f(x_n)|)$ converges. [2]

Solution: Find (x_n) such that $\cdots \leq |f(x_n)| \cdots \leq |f(x_2)| \leq |f(x_1)|$. [1]

Since $(|f(x_n)|)$ is decreasing and bounded below by 0, it converges. [1]