Indian Institute of Technology Kanpur

QUIZ-2, 23-11-2020
MTH-101A

(1) Let $A : \{(x, y) \in \mathbb{R}^2 : x \geq 0 \ and \ y \in \mathbb{R}\}$ and $f : A \to \mathbb{R}$ be the function defined as

$$f(x,y) = \begin{cases} xy \ sin(xy) & \text{if } x \geq 0 \text{ and } y \geq 0 \\ -xy \ sin(xy) & \text{if } x \geq 0 \text{ and } y < 0 \end{cases}$$

Determine the points on A where f is continuous.

(2) Let $f(x,y) = \frac{1}{2}(||x| - |y|| - |x| - |y|) - xy$.

(a) Does the directional derivative of f exist at $(0,0)$ in the direction $(\frac{3}{5}, \frac{4}{5})$? Justify your answer.

(b) Is f differentiable at $(0,0)$? Justify your answer.

(3) Let S be the solid obtained by revolving the region bounded by the curves $f_1(x) = x^2 - 4$ and $f_2(x) = 4 - x^2$ about the line $x = 2$. Using the shell method determine the volume of the solid S.

Solutions:

(1) Case 1: Let $x = 0$ or $y = 0$.
Suppose $x_n \to x$ and $y_n \to y$.
Then $|f(x_n, y_n) - f(x, y)| = |f(x_n, y_n)| \leq |x_n y_n| \to 0$.
Hence f is continuous at (x, y). [2]
Case 2: Let $x > 0$ and $y > 0$.
Suppose $x_n \to x$ and $y_n \to y$.
Then $x_n > 0$ and $y_n > 0$ eventually and
$|f(x_n, y_n) - f(x, y)| = |x_n y_n sin(x_n y_n) - xysin(xy)| \to 0$.
Hence f is continuous at (x, y). [2]
Case 3: Let $x > 0$ and $y > 0$.
This case is similar to Case 1.
Therefore f is continuous at points of A. [1]

(2) Note that $\dfrac{f(t\frac{3}{5}, t\frac{4}{5}) - f(0,0)}{t} = \dfrac{|t|}{2t}\{| \, |\frac{3}{5}| - |\frac{4}{5}| \, | - |\frac{3}{5}| - |\frac{4}{5}|\} - t\frac{3}{5} \cdot \frac{4}{5}$.
Hence $\lim_{t \to 0} \dfrac{f(t\frac{3}{5}, t\frac{4}{5}) - f(0,0)}{t}$ does not exist. [3]
Since the directional directive of f at $(0,0)$ in a direction does not exist, f is NOT differentiable at $(0,0)$. [2]

(3) Volume $= \int_{-2}^{2} 2\pi(2-x)((4-x^2)-(x^2-4))dx$ [2+2]

Volume $= \frac{256\pi}{3}$. [1]