MTH 101 AA Quiz 2

Tentative Marking scheme

1. Discuss the convergence or divergence of the improper integral

$$\int_0^2 \frac{x(1+sin^2x)}{2-x}\,dx.$$

[4]

Solution :

Let $g(x) = \frac{1}{2-x}$ and $f(x) = \dfrac{x(1+sin^2x)}{2-x}$. [1]

Observe that $\lim_{x\to 2} \dfrac{f(x)}{g(x)} = 2(1+sin^2 2) \neq 0$. [2]

As $\int_0^2 \frac{1}{2-x}dx$ diverges, by the LCT, $\int_0^2 f(x)\,dx$ diverges. [1]

(Note that if $I = \int_0^2 \frac{1}{2-x}dx$, then $I = \int_0^2 \frac{1}{t}dt$)

OR

Note that $f(x) \geq \frac{x}{2-x} = h(x)$.

Observe that $\lim_{x\to 2} \dfrac{h(x)}{g(x)} = 2 \neq 0$.

As $\int_0^2 \frac{1}{2-x}dx$ diverges, by the LCT, $\int_0^2 h(x)\,dx$ diverges.

Hence $\int_0^2 f(x)\,dx$ diverges.

2. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that $\int_0^1 f(t)dt = 1$ and $f(0) = 0$. Show that there exists $c \in (0,1)$ such that $f'(c) = 2$. [5]

   Solution

   Let $F(x) = \int_0^x f(t)dt$. [2]

   By EMVT, $F(1) = F(0) + F'(0) + \frac{F''(c)}{2}$ for some $c \in (0,1)$ [2]

   Note that $F(1) = 1$, $F'(0) = f(0) = 0$, $F(0) = 0$ and $F''(x) = f'(x)$ [1]

3. Let R be the region bounded by the y-axis, the line $y = 2x$ and the curve $y = 3 - x^2$. Let S_1 be the solid obtained by revolving R about the x-axis and S_2 be the solid obtained by revolving R about the line $y = -1$. Find the volume of S_1 by the Washer Method and the volume of S_2 by the Shell Method. [6]

   Solution

The point of intersection of $y = 3 - x^2$ with the y- axis is $(0, 3)$ and the point of intersection of $y = 3 - x^2$ with the line $y = 2x$ is $(1, 2)$. [1]

Volume of $S_1 = \int_0^1 \pi((3 - x^2)^2 - (2x)^2)\ dx$ [1]

Volume of $S_1 = \dfrac{88\pi}{15}$. [1]

Volume of $S_2 = \int_0^2 2\pi(y + 1)\frac{y}{2}\ dy + \int_2^3 2\pi(y + 1)\sqrt{3 - y}\ dy$ [2]

Volume of $S_2 = \dfrac{46\pi}{5}$. [1]