Lectures 11 - 13: Infinite Series, Convergence tests, Leibniz's theorem

**Series**: Let \((a_n)\) be a sequence of real numbers. Then an expression of the form \(a_1 + a_2 + a_3 + \ldots\) denoted by \(\sum_{n=1}^{\infty} a_n\), is called a series.

**Examples**: 1. \(1 + \frac{1}{2} + \frac{1}{3} + \ldots\) or \(\sum_{n=1}^{\infty} \frac{1}{n}\)
2. \(1 + \frac{1}{4} + \frac{1}{9} + \ldots\) or \(\sum_{n=1}^{\infty} \frac{1}{n^2}\)

**Partial sums**: \(S_n = a_1 + a_2 + a_3 + \ldots + a_n\) is called the nth partial sum of the series \(\sum_{n=1}^{\infty} a_n\).

**Convergence or Divergence of** \(\sum_{n=1}^{\infty} a_n\)

If \(S_n \to S\) for some \(S\) then we say that the series \(\sum_{n=1}^{\infty} a_n\) converges to \(S\). If \((S_n)\) does not converge then we say that the series \(\sum_{n=1}^{\infty} a_n\) diverges.

**Examples**:
1. \(\sum_{n=0}^{\infty} \log(n+1)\) diverges because \(S_n = \log(n + 1)\).
2. \(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\) converges because \(S_n = 1 - \frac{1}{n+1} \to 1\).
3. If \(0 < x < 1\), then the geometric series \(\sum_{n=0}^{\infty} x^n\) converges to \(\frac{1}{1-x}\) because \(S_n = \frac{1-x^{n+1}}{1-x}\).

**Necessary condition for convergence**

**Theorem 1**: If \(\sum_{n=1}^{\infty} a_n\) converges then \(a_n \to 0\).

**Proof**: \(S_{n+1} - S_n = a_{n+1} \to 0\).

The condition given in the above result is necessary but not sufficient i.e., it is possible that \(a_n \to 0\) and \(\sum_{n=1}^{\infty} a_n\) diverges.

**Examples**:
1. If \(|x| \geq 1\), then \(\sum_{n=1}^{\infty} x^n\) diverges because \(a_n \to 0\).
2. \(\sum_{n=1}^{\infty} \sin n\) diverges because \(a_n \to 0\).
3. \(\sum_{n=1}^{\infty} \log(n+1)\) diverges, however, \(\log(n+1) \to 0\).

**Necessary and sufficient condition for convergence**

**Theorem 2**: Suppose \(a_n \geq 0\ \forall\ n\). Then \(\sum_{n=1}^{\infty} a_n\) converges if and only if \((S_n)\) is bounded above.

**Proof**: Note that under the hypothesis, \((S_n)\) is an increasing sequence.

**Example**: The Harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) diverges because
\[
S_{2^k} \geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \ldots + 2^{k-1} \cdot \frac{1}{2^k} = 1 + \frac{k}{2}
\]
for all \(k\).

**Theorem 3**: If \(\sum_{n=1}^{\infty} |a_n|\) converges then \(\sum_{n=1}^{\infty} a_n\) converges.

**Proof**: Since \(\sum_{n=1}^{\infty} |a_n|\) converges the sequence of partial sums of \(\sum_{n=1}^{\infty} |a_n|\) satisfies the Cauchy criterion. Therefore, the sequence of partial sums of \(\sum_{n=1}^{\infty} a_n\) satisfies the Cauchy criterion.

**Remark**: Note that \(\sum_{n=1}^{\infty} a_n\) converges if and only if \(\sum_{n=p}^{\infty} a_n\) converges for any \(p \geq 1\).
Tests for Convergence

Let us determine the convergence or the divergence of a series by comparing it to one whose behavior is already known.

**Theorem 4:** (Comparison test) Suppose \( 0 \leq a_n \leq b_n \) for \( n \geq k \) for some \( k \). Then

1. The convergence of \( \sum_{n=1}^{\infty} b_n \) implies the convergence of \( \sum_{n=1}^{\infty} a_n \).
2. The divergence of \( \sum_{n=1}^{\infty} a_n \) implies the divergence of \( \sum_{n=1}^{\infty} b_n \).

**Proof:** (1) Note that the sequence of partial sums of \( \sum_{n=1}^{\infty} a_n \) is bounded. Apply Theorem 2.

(2) This statement is the contrapositive of (1). \( \square \)

**Examples:**

1. \( \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \) converges because \( \frac{1}{(n+1)(n+2)} \leq \frac{1}{n(n+1)} \). This implies that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

2. \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges because \( \frac{1}{n} \leq \frac{1}{\sqrt{n}} \).

3. \( \sum_{n=1}^{\infty} \frac{1}{n!} \) converges because \( n^2 < n! \) for \( n \geq 4 \).

**Problem 1:** Let \( a_n \geq 0 \). Then show that both the series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} \frac{a_n}{a_{n+1}} \) converge or diverge together.

**Solution:** Suppose \( \sum_{n=1}^{\infty} a_n \) converges. Since \( 0 \leq \frac{a_n}{1+a_n} \leq a_n \) by comparison test \( \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \) converges.

Suppose \( \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \) converges. By the Theorem 1, \( \frac{a_n}{1+a_n} \to 0 \). Hence \( a_n \to 0 \) and therefore \( 1 + a_n < 2 \) eventually. Hence \( 0 \leq \frac{1}{2}a_n \leq \frac{a_n}{1+a_n} \). Apply the comparison test.

**Theorem 5:** (Limit Comparison Test) Suppose \( a_n, b_n \geq 0 \) eventually. Suppose \( \frac{a_n}{b_n} \to L \).

1. If \( L \in \mathbb{R}, L > 0 \), then both \( \sum_{n=1}^{\infty} b_n \) and \( \sum_{n=1}^{\infty} a_n \) converge or diverge together.
2. If \( L \in \mathbb{R}, L = 0 \), and \( \sum_{n=1}^{\infty} b_n \) converges then \( \sum_{n=1}^{\infty} a_n \) converges.
3. If \( L = \infty \) and \( \sum_{n=1}^{\infty} b_n \) diverges then \( \sum_{n=1}^{\infty} a_n \) diverges.

**Proof:** 1. Since \( L > 0 \), choose \( \epsilon > 0 \), such that \( L - \epsilon > 0 \). There exists \( n_0 \) such that \( 0 \leq L - \epsilon < \frac{a_n}{b_n} < L - \epsilon \). Use the comparison test.

2. For each \( \epsilon > 0 \), there exists \( n_0 \) such that \( 0 < \frac{a_n}{b_n} < \epsilon, \forall n > n_0 \). Use the comparison test.

3. Given \( \alpha > 0 \), there exists \( n_0 \) such that \( \frac{a_n}{b_n} > \alpha \forall n > n_0 \). Use the comparison test. \( \square \)

**Examples:**

1. \( \sum_{n=1}^{\infty} (1 - n \sin \frac{1}{n}) \) converges. Take \( b_n = \frac{1}{n^n} \) in the previous result.

2. \( \sum_{n=1}^{\infty} \frac{1}{n} \log(1 + \frac{1}{n}) \) converges. Take \( b_n = \frac{1}{n^n} \) in the previous result.

**Theorem 6** (Cauchy Test or Cauchy condensation test) If \( a_n \geq 0 \) and \( a_{n+1} \leq a_n \forall n \), then \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \sum_{k=0}^{\infty} 2^k a_{2k} \) converges.
Proof: Let $S_n = a_1 + a_2 + \ldots + a_n$ and $T_k = a_1 + 2a_2 + \ldots + 2^ka_{2^k}$.

Suppose $(T_k)$ converges. For a fixed $n$, choose $k$ such that $2^k \geq n$. Then

\[
S_n = a_1 + a_2 + \ldots + a_n \\
\leq a_1 + (a_2 + a_3) + \ldots + (a_{2^k} + \ldots + a_{2^k+1-1}) \\
\leq a_1 + 2a_2 + \ldots + 2^ka_{2^k} = T_k.
\]

This shows that $(S_n)$ is bounded above; hence $(S_n)$ converges.

Suppose $(S_n)$ converges. For a fixed $k$, choose $n$ such that $n \geq 2^k$. Then

\[
S_n = a_1 + a_2 + \ldots + a_n \\
\geq a_1 + a_2 + (a_3 + a_4) + \ldots + (a_{2^k-1+1} + \ldots + a_{2^k}) \\
\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \ldots + 2^{k-1}a_{2^k} = \frac{1}{2}T_k.
\]

This shows that $(T_k)$ is bounded above; hence $(T_k)$ converges. \qed

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
2. $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Problem 2: Let $a_n \geq 0, a_{n+1} \leq a_n$ for some $n$ and suppose $\sum a_n$ converges. Show that $na_n \to 0$ as $n \to \infty$.

Solution: By Cauchy condensation test $\sum_{k=0}^{\infty} 2^ka_{2^k}$ converges. Therefore $2^ka_{2^k} \to 0$ and hence $2^{k+1}a_{2^k} \to 0$ as $k \to \infty$. Let $2^k \leq n \leq 2^{k+1}$. Then $na_n \leq na_{2^k} \leq 2^{k+1}a_{2^k} \to 0$. This implies that $na_n \to 0$ as $n \to \infty$.

Theorem 7 (Ratio test) Consider the series $\sum_{n=1}^{\infty} a_n$, $a_n \neq 0 \forall n$.

1. If $\left|\frac{a_{n+1}}{a_n}\right| \leq q$ eventually for some $0 < q < 1$, then $\sum_{n=1}^{\infty} \left|a_n\right|$ converges.
2. If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ eventually then $\sum_{n=1}^{\infty} \left|a_n\right|$ diverges.

Proof: 1. Note that for some $N$, $|a_{n+1}| \leq q |a_n| \forall n \geq N$. Therefore, $|a_{N+p}| \leq q^p |a_N| \forall p > 0$. Apply the comparison test.

2. In this case $|a_n| \not\to 0$.

Corollary 1: Suppose $a_n \neq 0 \forall n$, and $\left|\frac{a_{n+1}}{a_n}\right| \to L$ for some $L$.

1. If $L < 1$ then $\sum_{n=1}^{\infty} \left|a_n\right|$ converges.
2. If $L > 1$ then $\sum_{n=1}^{\infty} \left|a_n\right|$ diverges.
3. If $L = 1$ we cannot make any conclusion.

Proof: 1. Note that $\left|\frac{a_{n+1}}{a_n}\right| < L + \frac{(1-L)}{2}$ eventually. Apply the previous theorem.
2. Note that \(| \frac{a_{n+1}}{a_n} \| > L - \frac{(L-1)}{2} \) eventually. Apply the previous theorem.

Examples :

1. \( \sum_{n=1}^{\infty} \frac{1}{n} \) converges because \( \frac{a_{n+1}}{a_n} \to 0 \).
2. \( \sum_{n=1}^{\infty} \frac{n}{n^2} \) diverges because \( \frac{a_{n+1}}{a_n} = (1 + \frac{1}{n})^n \to e > 1 \).
3. \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, however, in both these cases \( \frac{a_{n+1}}{a_n} \to 1 \).

**Theorem 8 : (Root Test )** If \( 0 \leq a_n \leq x^n \) or \( 0 \leq a_n^{1/n} \leq x \) eventually for some \( 0 < x < 1 \) then \( \sum_{n=1}^{\infty} |a_n| \) converges.

**Proof :** Immediate from the comparison test.

**Corollary 2:** Suppose \( |a_n|^{1/n} \to L \) for some \( L \). Then

1. If \( L < 1 \) then \( \sum_{n=1}^{\infty} |a_n| \) converges.
2. If \( L > 1 \) then \( \sum_{n=1}^{\infty} a_n \) diverges.
3. If \( L = 1 \) we cannot make any conclusion.

Examples :

1. \( \sum_{n=2}^{\infty} \frac{1}{(logn)^n} \) converges because \( a_{n/\log} = \frac{1}{\log n} \to 0 \).
2. \( \sum_{n=1}^{\infty} (\frac{n}{n+1})^n \) converges because \( a_{n/\log} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1 \).
3. \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, however, in both these cases \( a_{n/\log} \to 1 \).

**Theorem 9 : (Leibniz test )** If \( (a_n) \) is decreasing and \( a_n \to 0 \), then \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \) converges.

**Proof :** Note that \( (S_{2n}) \) is increasing and bounded above by \( S_1 \). Similarly, \( (S_{2n+1}) \) is decreasing and bounded below by \( S_2 \). Therefore both converge. Since \( S_{2n+1} - S_{2n} = a_{2n+1} \to 0 \), both \( (S_{2n+1}) \) and \( (S_{2n}) \) converge to the same limit and therefore \( (S_n) \) converges.

Examples : \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}, \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \) and \( \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \log n} \) converge.

**Problem 3:** Let \( \{a_n\} \) be a decreasing sequence, \( a_n \geq 0 \) and \( \lim_{n \to \infty} a_n = 0 \). For each \( n \in \mathbb{N} \), let \( b_n = \sum_{n=1}^{\infty} (-1)^n a_n \). Show that \( \sum_{n=1}^{\infty} (-1)^n b_n \) converges.

**Solution :** Note that \( b_{n+1} - b_n = \frac{1}{n+1} (a_1 + a_2 + ... + a_{n+1}) - \frac{1}{n} (a_1 + ... + a_n) = \frac{a_{n+1}}{n+1} - \frac{(a_1 + ... + a_n)}{n(n+1)} \).

Since \( (a_n) \) is decreasing, \( a_1 + ... + a_n \geq na_n \). Therefore, \( b_{n+1} - b_n \leq \frac{a_{n+1} - a_n}{n+1} \leq 0 \). Hence \( (b_n) \) is decreasing.

We now need to show that \( b_n \to 0 \). For a given \( \epsilon > 0 \), since \( a_n \to 0 \), there exists \( n_0 \) such that \( a_n < \frac{\epsilon}{2} \) for all \( n \geq n_0 \).

Therefore, \( \frac{a_{n+1} + a_n}{n} \leq \frac{a_{n+1} + a_n}{n} \leq \frac{a_{n+1} + a_n}{n} \leq \frac{\epsilon}{2} \). Choose \( N \geq n_0 \) large enough so that \( \frac{a_{n+1} + a_n}{n} < \frac{\epsilon}{2} \). Then, for all \( n \geq N \), \( \frac{a_{n+1} + a_n}{n} < \epsilon \). Hence, \( b_n \to 0 \). Use the Leibniz test for convergence.