Lecture 28: Directional Derivatives, Gradient, Tangent Plane

The partial derivative with respect to \( x \) at a point in \( \mathbb{R}^3 \) measures the rate of change of the function along the X-axis or say along the direction \((1,0,0)\). We will now see that this notion can be generalized to any direction in \( \mathbb{R}^3 \).

**Directional Derivative**: Let \( f : \mathbb{R}^3 \to \mathbb{R}, \; X_0 \in \mathbb{R}^3 \) and \( U \in \mathbb{R}^3 \) such that \( \| U \| = 1 \). The directional derivative of \( f \) in the direction \( U \) at \( X_0 = (x_0, y_0, z_0) \) is defined by

\[
D_{X_0}f(U) = \lim_{t \to 0} \frac{f(X_0 + tu) - f(X_0)}{t}
\]

provided the limit exists.

It is clear that \( D_{X_0}f(e_1) = f_x(X_0), \; D_{X_0}f(e_2) = f_y(X_0) \) and \( D_{X_0}f(e_3) = f_z(X_0) \).

The proof of the following theorem is similar to the proof of Theorem 26.2.

**Theorem 28.1**: If \( f \) is differentiable at \( X_0 \), then \( D_{X_0}f(U) \) exists for all \( U \in \mathbb{R}^3, \| U \| = 1 \). Moreover, \( D_{X_0}f(U) = f'(X_0) \cdot U = (f_x(X_0), f_y(X_0), f_z(X_0)) \cdot U \).

The previous theorem says that if a function is differentiable then all its directional derivatives exist and they can be easily computed from the derivative.

**Examples**:

(i) In this example we will see that a function is not differentiable at a point but the directional derivatives in all directions at that point exist.

Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f(x,y) = \frac{x^2y}{x^2+y^2} \) when \((x,y) \neq (0,0)\) and \( f(0,0) = 0 \).

This function is not continuous at \((0,0)\) and hence it is not differentiable at \((0,0)\).

We will show that the directional derivatives in all directions at \((0,0)\) exist. Let \( U = (u_1,u_2) \in \mathbb{R}^3, \; \| U \| = 1 \) and \( O = (0,0) \). Then

\[
\lim_{t \to 0} \frac{f(0 + tU) - f(0)}{t} = \lim_{t \to 0} \frac{t^3u_1^2u_2}{t(t^4u_1^4 + t^2u_2^2)} = \lim_{t \to 0} \frac{u_1^2u_2}{u_1^4 + u_2^2} = 0, \text{ if } u_2 = 0 \text{ and } \frac{u_1^2}{u_2}, \text{ if } u_2 \neq 0
\]

Therefore, \( D_0f((u_1,0)) = 0 \) and \( D_0f((u_1,u_2)) = \frac{u_1^2}{u_2} \) when \( u_2 \neq 0 \).

(ii) In this example we will see that the directional derivative at a point with respect to some vector may exist and with respect to some other vector may not exist.

Consider the function \( f(x,y) = \frac{x}{y} \) if \( y \neq 0 \) and 0 if \( y = 0 \). Let \( U = (u_1,u_2) \) and \( \| U \| = 1 \). It is clear that if \( u_1 = 0 \) or \( u_2 = 0 \), then \( D_0f(U) \) exists and is equal to 0. If \( u_1u_2 \neq 0 \) then

\[
\lim_{t \to 0} \frac{f(0 + tU) - f(0)}{t} = \lim_{t \to 0} \frac{u_1}{tu_2}
\]

does not exist. So, only the partial derivatives of the function at \( O \) exist. Note that this function can not be differentiable at \( O \) (Why?).

**Problem 1**: Let \( f(x,y) = \frac{x}{|y|} \sqrt{x^2 + y^2} \) if \( y \neq 0 \) and \( f(x,y) = 0 \) if \( y = 0 \). Show that \( f \) is continuous at \((0,0)\), it has all directional derivatives at \((0,0)\) but it is not differentiable at \((0,0)\).

**Solution**: Note that \( |f(x,y) - f(0,0)| = \sqrt{x^2 + y^2} \). Hence the function is continuous.
For \( \| (u_1, u_2) \| = 1, \) \( \lim_{t \to 0} \frac{f(tu_1, tu_2)}{t} = 0 \) if \( u_2 = 0 \) and \( \frac{u_2}{|u_2|} \) if \( u_2 \neq 0 \). Therefore directional derivatives in all directions exist.

Note that \( f_x(0, 0) = 0 \) and \( f_y(0, 0) = 1 \). If \( f \) is differentiable at \((0, 0)\) then \( f'(0, 0) = \alpha = (0, 1) \).

\[
\epsilon(h, k) = \frac{k}{|k|} \sqrt{h^2 + k^2} - k \sqrt{h^2 + k^2} \to 0 \text{ as } (h, k) \to (0, 0).
\]

For example, \( h = k \) gives \((\sqrt{2} - 1) \frac{k}{|k|} \to 0 \) as \( k \to 0 \). Therefore the function is not differentiable at \((0, 0)\). \( \square \)

The vector \((f_x(X_0), f_y(X_0), f_z(X_0))\) is called gradient of \( f \) at \( X_0 \) and is denoted by \( \nabla f(X_0) \).

**An Application:** Let us see an application of Theorem 1. Suppose \( f \) is differentiable at \( X_0 \). Then \( f'(X_0) = \nabla f(X_0) \) and \( D_{X_0}f(U) = \nabla f(X_0) \cdot U = \| \nabla f(X_0) \| \cos \theta \) where \( \theta \in [0, \pi] \) is the angle between the gradient and \( U \). Suppose \( \nabla f(X_0) \neq 0 \). Then \( D_{X_0}f(U) \) is maximum when \( \theta = 0 \) and minimum \( \theta = \pi \). That is, \( f \) increases (respectively, decreases) most rapidly around \( X_0 \) in the direction \( U = \frac{\nabla f(X_0)}{\| \nabla f(X_0) \|} \) (respectively, \( U = -\frac{\nabla f(X_0)}{\| \nabla f(X_0) \|} \)).

**Example:** Suppose the temperature of a metallic sheet is given as \( f(x, y) = 20 - 4x^2 - y^2 \). We will start from the point \((2, 1)\) and find a path i.e., a plane curve, \( r(t) = (x(t))i + y(t)j \) which is a path of maximum increase in the temperature. Note that the direction of the path is \( r'(t) \). This direction should coincide with that of the maximum increase of \( f \). Therefore, \( \alpha r'(t) = \nabla f \) for some \( \alpha \). This implies that \( \alpha x' = -8x \) and \( \alpha y' = -2y \). By chain rule we have \( \frac{dy}{dx} = \frac{\alpha y}{\alpha x} = \frac{2y}{4x} \). Since the curve passes through \((2, 1)\), we get \( x = 2y^4 \).

We will now see a geometric interpretation of the derivative i.e. gradient.

**Tangent Plane:** Suppose \( f : \mathbb{R}^3 \to \mathbb{R} \) is differentiable and \( c \in \mathbb{R} \). Consider the surface \( S = \{(x, y, z) : f(x, y, z) = c\} \). This surface is called a level surface at the height \( c \). (For example if \( f(x, y, z) = x^2 + y^2 + z^2 \) and \( c = 1 \), then \( S \) is the unit sphere.) Let \( P = (x_0, y_0, z_0) \) be a point on \( S \) and \( R(t) = (x(t), y(t), z(t)) \) be a differentiable (i.e., smooth) curve lying on \( S \). With these assumptions we prove the following result.

**Theorem 28.2:** If \( T \) is the tangent vector to \( R(t) \) at \( P \) then \( \nabla f(P) \cdot T = 0 \).

**Proof:** Since \( R(t) \) lies on \( S \), \( f(x(t), y(t), z(t)) = c \). Hence \( \frac{df}{dt} = 0 \). By chain rule,

\[
\frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} + \frac{df}{dz} \frac{dz}{dt} = 0 \quad \text{i.e.,} \quad \nabla f \cdot \frac{dR}{dt} = 0 \quad \text{i.e.,} \quad \nabla f \cdot T = 0 \text{ at } P. \quad \square
\]

From the previous theorem we conclude the following. Note that the gradient \( \nabla f(P) \) is perpendicular to the tangent vector to every smooth curve \( R(t) \) on \( S \) passing through \( P \). That is, all these tangent vectors lie on a plane which is perpendicular to \( \nabla f(P) \). That is, \( \nabla f(P) \), when \( \nabla f(P) \neq 0 \), is the normal to the surface at \( P \). Therefore, the plane through \( P \) with normal \( \nabla f(P) \) defined by

\[
f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0
\]

is called the tangent plane to the surface \( S \) at \( P = (x_0, y_0, z_0) \).

Suppose the surface is given as a graph of \( f(x, y) \), i.e., \( S = \{(x, y, f(x, y)) : (x, y) \in D \subseteq \mathbb{R}^2\} \). Then it can be considered as a level surface \( S = \{(x, y, z) : F(x, y, z) = 0\} \) where \( F(x, y, z) = f(x, y) - z \). Let \( X_0 = (x_0, y_0), z_0 = f(x_0, y_0) \) and \( P = (x_0, y_0, z_0) \). Then the equation of the tangent plane is \( f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0 \), i.e.,

\[
z = f(X_0) + f'(X_0)(X - X_0), \quad X = (x, y) \in \mathbb{R}^2.
\]