Lecture 29 : Mixed Derivative Theorem, MVT and Extended MVT

If \( f : \mathbb{R}^2 \to \mathbb{R} \), then \( f_x \) is a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \) (if it exists). So one can analyze the existence of

\[
 f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \quad \text{and} \quad f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)
\]

which are partial derivatives of \( f_x \) with respect \( x \) or \( y \) and, similarly the existence of \( f_{yy} \) and \( f_{yx} \). These are called second order partial derivatives of \( f \).

The following example shows that, in general, \( f_{xy} \) need not be equal to \( f_{yx} \).

**Example :** Let \( f(x, y) = xy - \frac{y^2}{x^2 + y^2} \) if \( (x, y) \neq (0, 0) \) and \( f(0, 0) = 0 \). Note that

\[
 f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k \to 0} \frac{h^2}{h^2 + k^2} = h
\]

and

\[
 f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = 1.
\]

Similarly, \( f_{xy}(0, 0) = -1 \).

**Theorem 29.1 (Mixed derivative theorem) :** If \( f(x, y) \) and its partial derivatives \( f_x, f_y, f_{xy} \) and \( f_{yx} \) are defined in a neighborhood of \( (x_0, y_0) \) and all are continuous at \( (x_0, y_0) \) then \( f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0) \).

We will not present the proof of this result here. The proof is given in the text book.

**Mean Value Theorem :** We will present the MVT for functions of several variables which is a consequence of MVT for functions of one variable.

**Theorem 29.2:** Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \) is differentiable. Let \( X_0 = (x_0, y_0) \) and \( X = (x_0 + h, y_0 + k) \). Then there exists \( C \) which lies on the line joining \( X_0 \) and \( X \) such that

\[
 f(X) = f(X_0) + f'(C)(X - X_0)
\]

i.e, there exists \( c \in (0, 1) \) such that

\[
 f(x_0 + h, y_0 + k) = f(x_0, y_0) + hf_x(C) + kf_y(C) \quad \text{where} \quad C = (x_0 + ch, y_0 + ck).
\]

**Proof :** Define \( \phi : [0, 1] \to \mathbb{R} \) by

\[
 \phi(t) = f(x_0 + th, y_0 + tk), \quad t \in [0, 1].
\]

Note that by the Chain Rule \( \phi \) is differentiable and

\[
 \phi' = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.
\]

By the MVT, there exist \( c \in (0, 1) \) such that

\[
 \phi(1) - \phi(0) = \phi'(c).
\]

This proves the result.

**Remark :** In the previous result if we fix \( X_0 \) and \( X \) then it is enough to assume that the function \( f \) is differentiable on the line segment joining \( X \) and \( X_0 \).
Problem: If $f(x, y)$ is constant if and only if $f_x = 0$ and $f_y = 0$.

We will now take up the extended mean value theorem which we need.

**Theorem 29.3 (EMVT):** Let $f, X, X_0$ be as in the previous theorem. Suppose $f_x$ and $f_y$ are continuous and they have continuous partial derivatives. Then, there exists $C$ which lies on the line joining $X_0$ and $X$ such that

$$f(X) = f(X_0) + f'(X_0)(X - X_0) + \frac{1}{2}(X - X_0)f''(C)(X - X_0)$$

where $f'' = \left( \begin{array}{cc} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{array} \right)$. That is, there exists $c \in (0, 1)$ such that

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + (hf_x + kf_y)(X_0) + \frac{1}{2}(h^2f_{xx} + 2hkf_{xy} + k^2f_{yy})(C)$$

where $C = (x_0 + ch, y_0 + ck)$.

**Proof (*):** Consider the function $\phi(t)$ defined in the proof of the previous result. Since $f_x$ and $f_y$ are continuous $f$ is differentiable. Therefore, as given in the proof of the previous theorem, $\phi$ is differentiable and

$$\phi' = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$ 

Since $f_x$ and $f_y$ have continuous partial derivatives, they are differentiable. Denote

$$\phi'(t) = hf_x(x_0 + th, y_0 + tk) + kf_y(x_0 + th, y_0 + tk) = F(x_0 + th, y_0 + tk), \ t \in [0, 1].$$

Again by the Chain Rule,

$$\phi'' = hF_x + kF_y = h \frac{\partial}{\partial x}(hf_x + kf_y) + k \frac{\partial}{\partial y}(hf_x + kf_y) = h(h \frac{\partial^2 f}{\partial x^2} + k \frac{\partial^2 f}{\partial y \partial x}) + k(h \frac{\partial^2 f}{\partial x \partial y} + k \frac{\partial^2 f}{\partial y^2}).$$

By the mixed derivative theorem,

$$\phi'' = h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}.$$ 

By the EMVT for $\phi$, there exists $c \in (0, 1)$ such that

$$\phi(1) = \phi(0) + \phi'(0) + \frac{\phi''(c)}{2}.$$ 

By substituting $\phi, \phi'$ and $\phi''$ in the above equation we get the result. 

**Remarks:** 1. We will consider $f''$, given in the statement of the previous theorem, as a notation. We do not say that the function $f$ is twice differentiable.

2. We will recall the EMVT when we will deal with the second derivative test for local maxima and minima of $f(x, y)$ in the next lecture.

3. Whatever we discussed above can be generalized to the functions of three variables.

4. The matrix given in the statement of the previous theorem is called Hessian matrix. We should be able to guess what should be the corresponding Hessian matrix for the functions of three variables.

5. Note that we applied the MVT and the EMVT for the function $\phi$ to get the MVT and the EMVT for $f(x, y)$. Similarly by assuming that $f(x, y)$ has continuous partial derivatives of order $n$ and applying Taylor’s theorem for the function $\phi$, we can obtain Taylor’s Theorem for $f(x, y)$. 