Lecture 4 : Continuity and limits

Intuitively, we think of a function $f : \mathbb{R} \to \mathbb{R}$ as continuous if it has a continuous curve. The term *continuous curve* means that the graph of f can be drawn without *jumps*, i.e., the graph can be drawn with a *continuous* motion of the pencil without leaving the paper.

Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has a discontinuous graph as shown in the following figure.

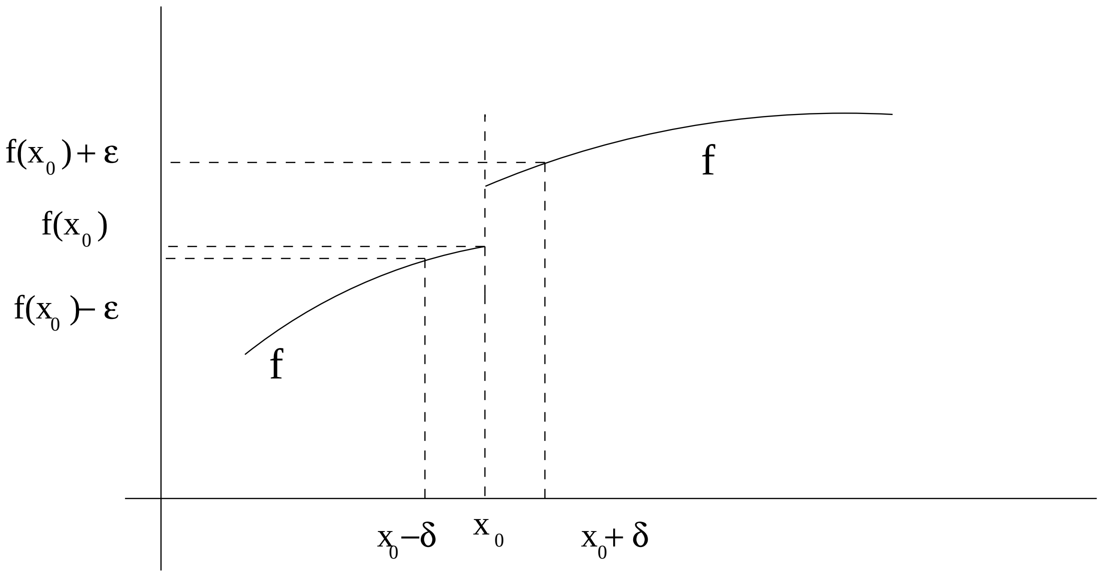

Figure 1: Discontinuous Graph

The graph is broken at the point $(x_0, f(x_0))$, i.e., the function f is discontinuous at x_0. Hence whenever x is close to x_0 from the right, $f(x)$ does not get close to $f(x_0)$. (The idea of getting close has already been discussed while dealing with convergent sequences). As shown in the figure, we can choose a neighbourhood $(f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$, $\epsilon_0 > 0$, at $f(x_0)$ such that if we take **any** neighbourhood $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$, then the image of the interval $(x_0 - \delta, x_0 + \delta)$ does not lie inside $(f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$. In formal terms, **there exists** $\epsilon > 0$ such that **for all** $\delta > 0$, $|x - x_0| < \delta \not\Rightarrow |f(x) - f(x_0) < \epsilon$. Hence if a function f is not continuous at x_0, we have the above condition.

We will now give the formal definition of continuity of a function at a point (in the "ϵ-δ language").

Definition A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at a point $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_o| < \delta$.

Using the (visible) discontinuity in the above example, we were able to find some ϵ for which it was not possible to find any δ as in the definition. Roughly, f is continuous at x_0 if whenever x approaches x_0, $f(x)$ approaches $f(x_0)$. In some cases when f is not continuous at x_0, there may be a number A such that whenever x approaches x_0, $f(x)$ approaches A. In this case we call such a number A the limit of f at x_0. Formally, we have:

Definition : A number A is called the limit of a function f at a point x_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - A| < \epsilon$ whenever $0 < |x - x_0| < \delta$. If such a number A exists then it is unique.

In this case we write $\lim_{x \to x_0} f(x) = A$. It is clear that $f(x_0)$ is the limit of f at x_0 if f is continuous at x_0.

The reader is advised to see the strong analogy between the definition of limit point and the definition of convergence of sequence. Let us now characterize the continuity of a function at a point in terms of sequences.

Theorem 4.1 : *A real valued function f is continuous at $x_0 \in \mathbb{R}$ if and only if whenever a sequence of real numbers (x_n) converges to x_0, then the sequence $(f(x_n))$ converges to $f(x_0)$.*

Proof: Suppose f is continuous at x_0 and $x_n \to x_0$. Let us show that $f(x_n) \to f(x_0)$. Let $\epsilon > 0$ be given. We must find N such that $|f(x_n) - f(x_0)| < \epsilon$ for all $n \geq N$. Since f is continuous at x_0, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Since $x_n \to x_0$, there exists N such that $|x_n - x_0| < \delta$ for all $n \geq N$. This N serves our purpose.

To prove the converse, let us assume the contrary that f is not continuous at x_0. Then for some $\epsilon > 0$ and for each n, there is an element x_n such that $|x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - f(x_0)| \geq \epsilon$. This contradicts the fact that $x_n \to x_0$ implies $f(x_n) \to f(x_0)$. $\square$

Remark : To define the continuity of a function f at a point x_0, the function f has to be defined at x_0. But even if the function is not defined at x_0, one can define the limit of a function at x_0.

The proof of the following theorem is similar to the proof of the previous theorem.

Theorem 4.2: $\lim\limits_{x \to x_0} f(x) = A$ *if and only if whenever a sequence of real numbers (x_n) converges to x_0, $x_n \neq x_0$ for all n, then the sequence $(f(x_n))$ converges to A.*

Examples : 1. Define a function $f(x)$ such that $f(x) = 2x sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$. We will show that f is continuous at 0 using first by the $\varepsilon - \delta$ definition and then by the sequential characterization.

Using the $\varepsilon - \delta$ definition : Remember that for a given $\varepsilon > 0$, we have to find a $\delta > 0$ (not the other way!). Note that here $x_0 = 0$ and

$$| f(x) - f(x_0) | = | 2x sin(\frac{1}{x}) - 0 | \leq | 2x | = 2 | x - x_0 | .$$

Suppose that ε is given. Choose any $\delta > 0$ such that $\delta \leq \frac{\varepsilon}{2}$. Then we have

$$| f(x) - f(x_0) | < \varepsilon \text{ whenever } \quad | x - x_0 | < \delta.$$

This shows that f is continuous at $x_0 = 0$.

Using the sequential characterization : Note that $| f(x) | \leq 2 | x |$. Therefore, $f(x_n) \to f(0)$ whenever $x_n \to 0$. This proves that f is continuous at 0.

2. The function $f(x) = \sin(1/x)$ is defined for all $x \neq 0$. This function has no limit as $x \to 0$ because if we take $x_n = 2/\{\pi(2n+1)\}$ for $n = 1, 2, \ldots$, then $x_n \to 0$ but $f(x_n) = (-1)^n$ which does not tend to any limit as $n \to \infty$.

3. Let $f(x) = 0$ when x is rational and $f(x) = x$ when x is irrational. We will see that this function is continuous only at $x = 0$. Let (x_n) be any sequence such that $x_n \to 0$. Because, $| f(x_n) | \leq | x_n |$, $f(x_n) \to f(0)$. Therefore f is continuous at 0.

Suppose $x_0 \neq 0$ and it is rational. We will show that f is not continuous at x_0. Choose (x_n) such that $x_n \to x_0$ and all $x_n's$ are irrational numbers. Then $f(x_n) = x_n \to x_0 \neq f(x_0)$.This proves that f is not continuous at x_0. When x_0 is irrational, the proof is similar.

Remark : In order to show that a function is not continuous at a point x_0 it is sufficient to produce one sequence (x_n) such that $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$. However, to show a function is continuous at x_0, we have to show that $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$ i.e, for every (x_n) such that $x_n \to x_0$.

Continuous function on a subset of $\mathbb{R}$: Let S be a subset of $\mathbb{R}$ and $x_0 \in S$, we say that f is continuous at x_0, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$. Moreover, if f is continuous at each $x \in S$, then we say that f is continuous on S.

Limits at Infinity : Let $f : \mathbb{R} \to \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = A$ if for every $\epsilon > 0$, there exist $N > 0$ such that whenever $x \geq N$, we have $|f(x) - A| < \epsilon$.

Let $x_0 \in \mathbb{R}$. We say that $\lim_{x \to x_0} f(x) = \infty$ if for every M, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have $f(x) > M$.

Problem 1: *Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in \mathbb{R}$, $\mid f(x) - f(y) \mid \leq \mid x - y \mid$. Show that f is continuous.*

Solution : Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. Since $\mid f(x_n) - f(x_0) \mid \leq \mid x_n - x_0 \mid, f(x_n) \to f(x_0)$. Therefore f is continuous at x_0. Since x_0 is arbitrary, f is continuous everywhere.

Problem 2: *Let $f : (-1, 1) \to \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where f takes the value 0. Show that $f(0) = 0$.*

Solution : For every n, there exists $x_n \in (-\frac{1}{n}, \frac{1}{n})$ such that $f(x_n) = 0$. Since f is continuous at 0 and $x_n \to 0$, we have $f(x_n) \to f(0)$. Therefore, $f(0) = 0$.

Problem 3: *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every point $c \in \mathbb{R}$.*

Solution : First note that $f(0) = 0, f(-x) = -f(x)$ and $f(x - y) = f(x) - f(y)$. Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. Then $f(x_n) - f(x_0) = f(x_n - x_0) \to f(0) = 0$ as f is continuous at 0 and $x_n - x_0 \to 0$.

Properties of Continuous Functions on a Closed Interval :

Definition : Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$. We say that f is bounded on S if the set $f(S) := \{f(x) : x \in S\}$ is a bounded subset of $\mathbb{R}$.

We will now see some properties of continuous functions on a closed interval.

Theorem 4.3 : *If a function f is continuous on $[a, b]$ then it is bounded on $[a, b]$.*

Proof: Suppose that f is not bounded on $[a, b]$. Then for each natural number n there is a point $x_n \in [a, b]$ such that $|f(x_n)| > n$. Since (x_n) is a bounded sequence, by Bolzano-Weierstrass theorem it has a convergent subsequence, say $x_{n_k} \to x_0 \in [a, b]$. By the continuity of f, we have $f(x_{n_k}) \to f(x_0)$. This contradicts the assumption that $|f(x_n)| > n$ for all n. Hence f is bounded on $[a, b]$. □

We remark that if a function is continuous on an open interval (a, b) or on a semi-open interval of the type $(a, b]$ or $[a, b)$, then it is not necessary that the function has to be bounded. For example, consider the continuous function $\frac{1}{x}$ on $(0, 1]$.