Lecture 22: Areas of surfaces of revolution, Pappus’s Theorems

Let \( f : [a, b] \to \mathbb{R} \) be continuous and \( f(x) \geq 0 \). Consider the curve \( C \) given by the graph of the function \( f \). Let \( S \) be the surface generated by revolving this curve about the x-axis. We will define the surface area of \( S \) in terms of an integral expression.

Consider a partition \( P : a = x_0 < x_1 < x_2 < \ldots < x_n = b \) and consider the points \( P_i = (x_i, f(x_i)), i = 0, 1, 2, \ldots, n \). Join these points by straight lines as shown in Figure 1. Consider the segment \( P_{i-1}P_i \). The area \( A \) of the surface generated by revolving this segment about the x-axis is \( \pi(f(x_{i-1}) + f(x_i))\ell_i \) where \( \ell_i \) is the length of the segment \( P_{i-1}P_i \). This can be verified as follows. Note that the area \( A = \pi f(x_i)\alpha f(x_i) - \pi f(x_{i-1})\alpha f(x_{i-1}) \) (see Figure 2). Since

\[
\frac{\ell}{f(x_{i-1})} = \frac{\ell + \ell_i}{f(x_i) - f(x_{i-1})} = \alpha
\]

for some \( \alpha \), the area

\[
A = \pi f(x_i)\alpha f(x_i) - \pi f(x_{i-1})\alpha f(x_{i-1}) = \pi \alpha f(x_i) + f(x_{i-1}) (f(x_i) - f(x_{i-1})) = \pi \ell_i (f(x_{i-1}) + f(x_i)).
\]

The sum of the areas of the surfaces generated by the line segments is

\[
\sum_{i=1}^{n} \pi f(x_{i-1}) + f(x_i)\ell_i = \sum_{i=1}^{n} \pi f(x_{i-1}) \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} + \sum_{i=1}^{n} \pi f(x_i) \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}
\]

where \( \Delta y_i = f(x_i) - f(x_{i-1}) \). If \( f' \) is continuous, one can show that each of the sum given in the RHS of the above equation converges to \( \int_{a}^{b} \pi f(x) \sqrt{1 + (f'(x))^2} \, dx \) as \( \| P \| \to 0 \). In view of this we define the surface area generated by revolving the curve about the x-axis to be

\[
\int_{a}^{b} 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx.
\]

In case \( f(x) \leq 0 \), the formula for the area is \( \int_{a}^{b} 2\pi | f(x) | \sqrt{1 + (f'(x))^2} \, dx \).

**Example:** Let us find the area of the surface generated by revolving the curve \( y = \frac{1}{2}(x^2 + 1), 0 \leq x \leq 1 \) about the y-axis. Here the function \( y \) is increasing hence it is one-one and onto. Hence we can
write \( x \) in terms of \( y \): \( x = g(x) = \sqrt{2y - 1} \). In this case the formula is 
\[
\int_a^b 2\pi |g(y)| \sqrt{1 + (g'(y))^2} \, dy
\]
where \( a = 1/2 \) and \( b = 1 \).

**Parametric case:** If the curve is given in the parametric form \( \{(x(t), y(t)) : t \in [a, b]\} \), and \( x' \) and \( y' \) are continuous, then the surface area generated is
\[
\int_a^b 2\pi \rho(t) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt
\]
where \( \rho(t) \) is the distance between the axis of revolution and the curve.

**Example:** The curve \( x = t + 1, \ y = t^2 + t, \ 0 \leq t \leq 4 \) is rotated about the y-axis. Let us find the surface area generated. The surface area is
\[
\int_0^4 2\pi |t + 1| \sqrt{1 + (1 + t)^2} dt.
\]

**Polar case:** If the curve is given in the polar form, the surface area generated by revolving the curve about the x-axis is
\[
\int_a^b 2\pi y \sqrt{r^2 + (dr/d\theta)^2} \, d\theta = \int_a^b 2\pi r(\theta) \sin \theta \sqrt{r^2 + (dr/d\theta)^2} \, d\theta.
\]

**Example:** The lemniscate \( r^2 = 2a^2 \cos 2\theta \) is rotated about the x-axis. Let us find the area of the surface generated. A simple calculation shows that 
\[
\sqrt{r^2 + (dr/d\theta)^2} = 2a^2 / r.
\]
The curve is given in the notes of the previous lecture. The surface area is 
\[
2 \int_0^{\pi/4} 2\pi \sin 2\theta \, d\theta = 8\pi a^2 (1 - 1/\sqrt{2}).
\]

**Pappus’s Theorems:** There are two results of Pappus which relate the centroids to surfaces and solids of revolutions. The first result relates the centroid of a plane region with the volume of the solid of revolution generated by it.

**Theorem:** Let \( R \) be a plane region. Suppose \( R \) is revolved about the line \( L \) which does not cut through the interior of \( R \), then the volume of the solid generated is
\[
V = 2\pi \rho A
\]
where \( \rho \) is the distance from the axis of revolution to the centroid and \( A \) is the area of the region \( R \) (see Figure 3).

Note that in the above formula \( 2\pi \rho \) is the distance traveled by the centroid during the revolution. The second result relates the centroid of a plane curve with the area of the surface of revolution generated by the curve.

**Theorem:** Let \( C \) be a plane curve. Suppose \( C \) is revolved about the line \( L \) which does not cut through the interior of \( C \), then the area of the surface generated is
\[
S = 2\pi \rho L
\]
where \( \rho \) is the distance from the axis of revolution to the centroid and \( L \) is the length of the curve \( C \) (see Figure 3).

**Example:** Use a theorem of Pappus to find the centroid of the semi circular arc \( y = \sqrt{r^2 - x^2}, \ -r \leq x \leq r \). If the arc is revolved about the line \( y = r \), find the volume of the surface area generate.

**Solution:** We know the surface area generated by the curve \( 4\pi r^2 \) (see Figure 4). Let the centroid of the curve be \( (0, \overline{y}) \). By Pappus theorem \( 4\pi r^2 = 2\pi \overline{y} \pi r \) which implies that \( \overline{y} = \frac{2r}{\pi} \). Again by Pappus theorem, the area of the surface generated by revolving the curve around \( y = r \) is 
\[
2\pi (r - \overline{y}) \pi r = 2\pi r^2 (\pi - 2).
\]