Lecture 23: Review of vectors, equations of lines and planes; Sequences in $\mathbb{R}^3$

In the next two lectures we will deal with the functions from $\mathbb{R}$ to $\mathbb{R}^3$. Such functions are called vector valued functions. After two lectures we will deal with the functions of several variables, that is, functions from $\mathbb{R}^3$ or $\mathbb{R}^n$ to $\mathbb{R}$. Before discussing about the functions let us see some properties of $\mathbb{R}^3$. We first review some basic concepts from vector algebra.

Norm of a vector: If $X = (x, y, z)$, then the norm of X, denoted by $\|X\|$, is $\sqrt{x^2 + y^2 + z^2}$. $\|X - Y\|$ is the distance between the points X and Y.

Scalar product of two vectors: If $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$, then the scalar product of X and Y is $X \cdot Y = x_1y_1 + x_2y_2 + x_3y_3$.

Projection of a vector: The projection of a vector A along the non-zero vector B is $\frac{A \cdot B}{B \cdot B}B$.

Angle between two vectors : If θ is the angle between two vectors A and B then $A \cdot B = \|A\|\|B\| \cos\theta$.

Parametric and Cartesian equations of straight lines: The parametric representation of the straight line passing through P and parallel to a (non-zero) vector is $X - P = tA,\ t \in \mathbb{R}.$. If $X = (x, y, z),\ P = x_0, y_0, z_0)$ and $A = (a, b, c)$, then the above equation becomes

$$x = x_0 + ta,\ y = y_0 + tb \ \text{ and } \ z = z_0 + tc.$$

In case $a, b, c \neq 0$, then the equation of the line is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If $a = 0$, then the line is represented as $x = x_0$ and $\frac{y - y_0}{b} = \frac{z - z_0}{c}$.

Equation of a plane (passing through a point and perpendicular to a vector): The set of points $\{X : (X - P) \cdot N\} = \{X : X \cdot N = P \cdot N\}$ in $\mathbb{R}^3$ is the plane perpendicular to the vector N and passing the point P (see Figure 1). If $N = (a, b, c)$ and $P = (x_0, y_0, z_0)$ then the equation of the plane is

$$(x, y, z) \cdot (a, b, c) = (x_0, y_0, z_0) \cdot (a, b, c),$$

that is, $ax + by + cz = ax_0 + by_0 + cz_0$.

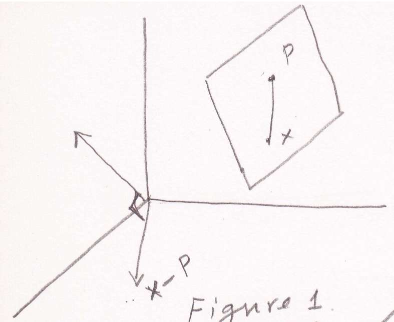

Figure 1

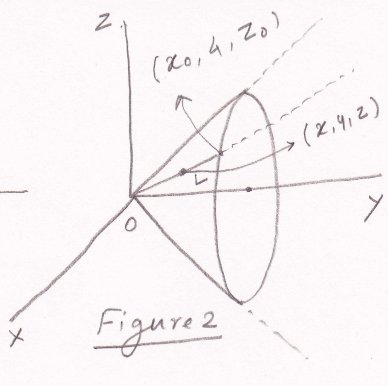

Figure 2

We can also derive equations of certain surfaces similar to the way we derived the equation of a plane.

Example: *Find the equation of the right circular cone having vertex at the origin and passing through the circle* $x^2 + y^2 = 25, \quad y = 4.$.

Solution: Let (x, y, z) be any arbitrary point on the surface. Let L be the straight line passing through (x, y, z) and $(0, 0, 0)$. Let $(x_0, 4, z_0)$ be the point of intersection of the line and the circle (see Figure 2). The equation of the line L is $\frac{x}{x_0} = \frac{y}{4} = \frac{z}{z_0}$. This implies that $x_0 = 4x/y$ and $z_0 = 4z/y$. Since x_0 and z_0 satisfy the equation of the circle, we have $4^2(\frac{x}{y})^2 + 4^2(\frac{z}{y})^2 = 25$. This implies that $16(x^2 + z^2) = 25y^2$.

Problem : *Determine the equation of the cylinder generated by a line through the curve* $(x-2)^2 + y^2 = 4,\ z = 0$ *moving parallel to the vector* $\vec{i} + \vec{j} + \vec{k}$.

Solution: Any point on the curve is of the form $(x_0, y_0, 0)$. The equation of a line passing through $(x_0, y_0, 0)$ and parallel to $(1, 1, 1)$ is $\frac{x-x_0}{1} = \frac{y-y_0}{1} = \frac{z}{1}$. We get $x_0 = x - z$ and $y_0 = y - z$. Since $(x_0, y_0, 0)$ lies on the curve, we get the equation of the cylinder to be $(x - z - 2)^2 + (y - z)^2 = 4$.

Convergence of a sequence in $\mathbb{R}^3$: We will see that the concept of convergence of sequence in $\mathbb{R}^3$ plays a role in studying about the vector valued functions and functions of several variables.

Let $X_n = (x_{1,n}, x_{2,n}, x_{3,n}) \in \mathbb{R}^3$. We say that the sequence (X_n) is convergent if there exists $X_0 \in \mathbb{R}^3$ such that $\|X_n - X_0\| \to 0$ as $n \to \infty$. In this case we say that X_n converges to X_0 and we write $X_n \to X_0$.

Note that corresponding to a sequence (X_n), $X_n = (x_{1,n}, x_{2,n}, x_{3,n})$, there are three sequences $(x_{1,n})(x_{2,n})$ and $(x_{3,n})$ in $\mathbb{R}$, and vice-versa. We will see that the properties of (X_n) can be completely understood in terms of the properties of the corresponding sequences $(x_{1,n})(x_{2,n})$ and $(x_{3,n})$ in $\mathbb{R}$.

Theorem 1. $X_n \to X_0$ *in* $\mathbb{R}^3 \Leftrightarrow$ *the coordinates* $x_{i,n} \to x_{i,0}$ *for every* $i = 1, 2, 3$ *in* $\mathbb{R}$.

Proof: This follows from the fact that $\sum_{i=1}^{3} |x_{i,n} - x_{i,0}|^2 \to 0 \Leftrightarrow |x_{i,n} - x_{i,0}| \to 0,\ i = 1, 2, 3.$ □

The proof of the following result is similar to the proof of the previous result.

Theorem 2. (X_n) is bounded (i.e., $\exists\ M$ such that $\|X_n\| \leq M\ \forall\ n$) $\Leftrightarrow$ each sequence $(x_{i,n}), i = 1, 2, 3$, is bounded.

Problem 1: *Every convergent sequence in* $\mathbb{R}^3$ *is bounded.*

Proof: If $\|X_n - X_0\| \to 0$, then $(\|X_n - X_0\|)$ is bounded. This implies that $(\|X_n\|)$ is bounded and this proves the result. □

Problem 2 (Bolzano-Weierstrass Theorem): *Every bounded sequence in* $\mathbb{R}^2$ *has a convergent subsequence.*

Proof (*): Suppose (x_n, y_n) be a bounded sequence. By Theorem 2 both (x_n) and (y_n) are bounded. By B-W theorem (x_n) has a convergent subsequence, say $x_{n_k} \to x_0$. Consider the sequence (y_{n_k}) and note that this sequence is also bounded. Again by B-W theorem, this sequence has a convergent subsequence, say $y_{n_{k_i}} \to y_0$. It is clear that the subsequence $(y_{n_{k_i}}, y_{n_{k_i}})$ of (x_n, y_n) converges to (x_0, y_0). □.

It is evident that the above theorem can also be extended to $\mathbb{R}^3$.