Lecture 34: Change of Variable in a Triple Integral; Area of a Parametric Surface

The change of variable formula for a double integral can be extended to triple integrals. We will straightaway present the formula.

**Formula:** \( \iiint_S f(x, y, z) dx dy dz = \iiint_T f(X(u, v, w), Y(u, v, w), Z(u, v, w)) | J(u, v, w) | du dv dw \)

where the Jacobian determinant \( J(u, v, w) \) is defined as follows:

\[
J(u, v, w) = \left| \begin{array}{ccc}
\frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\
\frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \\
\frac{\partial X}{\partial w} & \frac{\partial Y}{\partial w} & \frac{\partial Z}{\partial w}
\end{array} \right|.
\]

The above formula is valid under some assumptions which are similar to the assumptions we had for the two dimensional case.

**Special cases: 1. Cylindrical coordinates.** In this case the variables \( x, y \) and \( z \) are changed to \( r, \theta \) and \( z \) by the following three equations:

\[
x = X(r, \theta) = r \cos \theta, \quad y = Y(r, \theta) = r \sin \theta \quad \text{and} \quad z = z.
\]

We assume that \( r > 0 \) and \( \theta \) lies in \([0, 2\pi)\) or \( \theta_0 \leq \theta < \theta_0 + 2\pi \) for some \( \theta_0 \) as in the double integral case. We have basically replaced \( x \) and \( y \) by their polar coordinates in the \( xy \) plane and left \( z \) unchanged. The Jacobian is

\[
J(u, v, z) = \begin{vmatrix}
\cos \theta & \sin \theta & 0 \\
-r \sin \theta & r \cos \theta & 0 \\
0 & 0 & 1
\end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r.
\]

Therefore the change of variable formula is \( \iiint_S f(x, y, z) dx dy dz = \iiint_T f(r \cos \theta, r \sin \theta, z) r dr d\theta dz \).

**Example 1:** Let us evaluate \( \iiint_D (z^2 x^2 + z^2 y^2) dx dy dz \) where \( D \) is the region determined by \( x^2 + y^2 \leq 1, -1 \leq z \leq 1 \). Note that we can describe \( D \) in cylindrical coordinates: \( 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, -1 \leq z \leq 1 \). Therefore,

\[
\iiint_D (z^2 x^2 + z^2 y^2) dx dy dz = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^1 (z^2 r^2) r dr dz = \frac{1}{2\pi} \int_{-1}^1 z^2 dV = \frac{4}{3}.
\]

**2 Spherical Coordinates:** Suppose \((x, y, z)\) be a point \( \mathbb{R}^3 \). We will represent this point in terms of spherical coordinates \((r, \theta, \phi)\). The coordinates \( r, \theta \) and \( \phi \) are defined below.

Given a point \((x, y, z)\), let \( r = \sqrt{x^2 + y^2 + z^2} \) and \( \phi \) is the angle that the position vector \( xi + yj + zk \) makes with the (positive side of the) \( z \)-axis. The coordinate of \( z \) is given by \( z = r \cos \phi \).

To represent \( x \) and \( y \) in terms of spherical coordinates, represent \( x \) and \( y \) by polar coordinates in the \( xy \)-plane: \( x = r \cos \theta \) and \( y = r \sin \theta \). Since \( r = \rho \sin \phi \), the point \((x, y, z)\) is represented in terms of the spherical coordinates \((r, \theta, \phi)\) as follows:

\[
x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi.
\]

We keep \( r > 0, 0 \leq \theta < 2\pi \) and \( 0 \leq \phi < \pi \) to get a one-one mapping. The Jacobian determinant is \( J(r, \theta, \phi) = r^2 \sin \phi \). Since \( \sin \phi \geq 0 \), we have \( |J(r, \theta, \phi)| = r^2 \sin \phi \) and the change of variable formula is

\[
\iiint_S f(x, y, z) dx dy dz = \iiint_T f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho d\theta d\phi.
\]

**Example 2:** Let \( D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4a^2, \quad z \geq a\} \). Let us evaluate \( \iiint_D (z^2 x^2 + z^2 y^2) r^2 dz dV \).

We will use the spherical coordinates to solve this problem. If we allow \( \phi \) to vary independently,
then \( \phi \) varies from 0 to \( \pi \) (see Figure 2). If we fix \( \phi \) and allow \( \theta \) to vary from 0 to \( 2\pi \) then we obtain a surface of a cone (see Figure 1). Since only a part of the cone is lying in the given region, for a fixed \( \phi \) and \( \theta \), \( \rho \) varies from \( a \sec \phi \) to \( 2a \) (see Figure 1). Therefore the integral is

\[
\int_0^{2\pi} \int_0^a \cos \phi \ | J(\rho, \theta, \phi) | \ d\rho d\phi = 2\pi \int_0^a (2a \sin \phi \cos \phi - a \sin \phi) d\phi = \frac{\pi a}{2}.
\]

**Parametric Surfaces:** We defined a parametric curve in terms of a continuous vector valued function of one variable. We will see that a continuous vector valued function of two variables is associated with a surface, called parametric surface.

Let \( T \) be a region in \( \mathbb{R}^2 \) and \( r(u, v) = X(u, v)i + Y(u, v)j + Z(u, v)k \) be a continuous function on \( T \). The range of \( r \), \( \{ r(u, v) : (u, v) \in T \} \) is called a parametric surface (with the parameter domain \( T \) and the parameters \( u \) and \( v \)). We assume that the map \( r \) is one-one in the interior of \( T \) so that the surface does not cross itself. Sometimes the surface defined by \( r(u, v) \) is also expressed as

\[
x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v)
\]

where \( (u, v) \in T \)

and the above equations are called parametric equations of the surface.

**Examples:**

1. For a constant \( a > 0 \), \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \pi \) the equations \( x = a \sin \phi \cos \theta \), \( y = a \sin \phi \sin \theta \), \( z = a \cos \phi \) represent a sphere. Here the parameters are \( \theta \) and \( \phi \).

2. For a fixed \( a \), \( \infty < t < \infty \), \( 0 \leq \theta \leq 2\pi \), the equations \( x = a \cos \theta \), \( y = a \sin \theta \), \( z = t \) represent a cylinder. Here the parameters are \( t \) and \( \theta \).

3. A cone is represented by \( r(u, v) = \rho \sin \alpha \cos \theta i + \rho \sin \alpha \sin \theta j + \rho \cos \alpha k \) where \( \rho \geq 0 \), \( 0 \leq \theta \leq 2\pi \) and \( \alpha \) is fixed. Here the parameters are \( \rho \) and \( \theta \).

**Area of a Parametric Surface:** Let \( S = r(u, v) \) be a parametric surface defined on a parameter domain \( T \). Suppose \( r_u \) and \( r_v \) are continuous on \( T \) and \( r_u \times r_v \) is never zero on \( T \). Then the area of \( S \), denoted by \( a(S) \), is defined by the double integral

\[
a(S) = \iint_T || \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} || \ du dv.
\]

The formula can be justified as follows. Consider a small rectangle \( \Delta A \) in \( T \) with the sides on the lines \( u = u_0 \), \( u = u_0 + \Delta u \), \( v = v_0 \) and \( v = v_0 + \Delta v \). Consider the corresponding patch in \( S \), that is \( r(\Delta A) \). Note that the sides of \( \Delta A \) are mapped to the boundary curves of the patch \( r(\Delta A) \) by the map \( r \). The vectors \( r_u(u_0, v_0) \) and \( r_v(u_0, v_0) \) are tangents to the boundary curves of \( r(\Delta A) \) meeting at \( r(u_0, v_0) \). We now approximate the surface patch \( r(\Delta A) \) by the parallelogram whose sites are determined by the vectors \( \Delta ur_u \) and \( \Delta vr_v \). The area of this parallelogram is \( | r_u \times r_v | \Delta u \Delta v \). This will lead to the Riemann sum corresponding to the double integral \( \iint_T || \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} || \ du dv \).