Lecture 9 : Sufficient Conditions for Local Maximum, Point of Inflection

In Lecture 6, we have seen a necessary condition for local maximum and local minimum. In this lecture we will see some sufficient conditions.

Sufficient Conditions for Local Maximum and Local Minimum

We will present sufficient conditions only for local maximum and the sufficient conditions for local minimum are similar. In the following results we assume $f : (a, b) \to \mathbb{R}$.

Theorem 9.1 : *Let $c \in (a, b)$ and f be continuous at c. If for some $\delta > 0$, f is increasing on $(c - \delta, c)$ and decreasing on $(c, c + \delta)$, then f has a local maximum at c.*

Proof : Choose any x_1 and x such that $c - \delta < x_1 < x < c$. Then $f(x_1) \leq f(x)$ and by the continuity of f at c we have

$$f(x_1) \leq \lim_{x \to c^-} f(x) = f(c).$$

Similarly, if $c < x_2 < c + \delta$ then $f(x_2) \leq \lim_{x \to c^+} f(x) = f(c)$. This proves the result. $\square$

Corollary 9.1 : *Let $c \in (a, b)$ and f be continuous at c. If*

$$f'(x) \geq 0 \text{ for all } x \in (c - \delta, c) \text{ and } f'(x) \leq 0 \text{ for all } x \in (c, c + \delta)$$

then f has a local maximum at c.

Proof : The proof is immediate from the previous result. $\square$

Corollary 9.2 : *Let $c \in (a, b)$. If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.*

Proof (*): Since $f''(c)$ exists, $f'(x)$ exists in a neighborhood of c. As $f''(c) < 0$, we have

$$f''(c) \;=\; \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} \;=\; \lim_{x \to c} \frac{f'(x)}{x - c} \;<\; 0.$$

Therefore there exists a $\delta > 0$ such that

$$\frac{f'(x)}{x - c} \;<\; 0 \text{ for all } x \in (c - \delta, c) \cup (c, c + \delta).$$

This implies that $f'(x) > 0$ for all $x \in (c - \delta, c)$ and $f'(x) < 0$ for all $x \in (c, c + \delta)$. Now apply the previous corollary. $\square$

Remark : The converses of the previous results are not true, i.e.,

(i) If f is continuous at c and f has a local maximum at c, then f need not be increasing on $(c - \delta, c)$ or decreasing on $(c, c + \delta)$ for any $\delta > 0$. (Take the example : $f(x) = -(x sin(1/x))^2$ if $x \neq 0$, $f(0) = 0$ and $c = 0$.)

(ii) If f has a maximum at c and f is twice differentiable at c, then $f''(c)$ need not be less than 0. (Consider the example $f(x) = -x^4$ and $c = 0$).

So, the conditions assumed in the previous results are sufficient but not necessary.

Example : Let $f(x) = \frac{1}{x^4 - 2x^2 + 7} = \frac{1}{(x^2 - 1)^2 + 6}$, $x \in \mathbb{R}$. Then $f'(x) = \frac{-(4x^3 - 4x)}{(x^4 - 2x^2 + 7)^2} = \frac{-4x(x-1)(x+1)}{(x^4 - 2x^2 + 7)^2}$ and $f'(x) = 0$ for $x = -1, 0, 1$. By Corollary 9.1 and the corresponding result for local minimum,

f has a local minimum at $x = 0$ and local maxima at $x = -1$ and $x = 1$. In this example, it would be complicated to compute the second derivative and apply the second derivative test (Corollary 9.2).

Convexity, Concavity and Point of Inflection

Definition : Let $f : (a, b) \to \mathbb{R}$ be differentiable. We say that f is convex (resp., concave) on (a, b) if f' is strictly increasing (resp., strictly decreasing) on (a, b).

It is clear that if f is twice differentiable on (a, b) and $f''(x) > 0$ for all $x \in (a, b)$ then f is convex. A similar result also holds for concavity.

Examples : The function $f(x) = x^2$ on any open interval (in fact on all of $\mathbb{R}$) and the function $f(x) = sinx$ on $(\pi, 2\pi)$ are convex functions. The function $f(x) = -x^2$ on any open interval and the function $f(x) = sinx$ on $(0, \pi)$ are concave functions.

Definition : Let $f : (a, b) \to \mathbb{R}$ be continuous at a point $c \in (a, b)$. The point c is said to be a point of inflection if there exists a $\delta > 0$ such that either

$$f \text{ is convex on } (c-\delta, c) \text{ and } f \text{ is concave on } (c, c+\delta)$$

or

$$f \text{ is concave on } (c-\delta, c) \text{ and } f \text{ is convex on } (c, c+\delta).$$

It is clear that if $f''(x) > 0 \;\; \forall \;\; x \in (c-\delta, c)$ and $f''(x) < 0 \; \forall \;\; x \in (c, c+\delta)$ for some δ or $f''(x) < 0 \;\; \forall \;\; x \in (c-\delta, c)$ and $f''(x) > 0 \;\; \forall \;\; x \in (c, c+\delta)$ then c is a point of inflection.

Necessary Condition for Point of Inflection

Theorem 9.2: *Let $c \in (a, b)$ and $f''(c)$ exist. If f has a point of inflection at c then $f''(c) = 0$.*

Proof (*): Assume that f' is strictly increasing on $(c-\delta, c)$ and is strictly decreasing on $(c, c+\delta)$ for some $\delta > 0$. Since $f''(c)$ exists,

$$f''(c) = \lim_{x \to c^-} \frac{f'(x) - f'(c)}{x - c} \geq 0.$$

Similarly $f''(c) = \lim_{x \to c^+} \frac{f'(x)-f'(c)}{x-c} \leq 0$. Therefore $f''(0) = 0$. □

Remark : It is possible that $f''(c) = 0$ at a point but c is not a point of inflection. For example, $f(x) = x^4$ and $c = 0$. It is also possible that $f''(c)$ may not exist but c could be a point of inflection. For example $f(x) = x^{1/3}$ and $c = 0$.

Sufficient Condition for Point of Inflection

Theorem 9.3: *Let $c \in (a, b)$. If $f''(c) = 0$ and $f'''(c) \neq 0$ then c is a point of inflection.*

Proof(*): The proof is similar to the proof of Corollary 9.2 and it is left as an exercise. □

Remark : It is possible that c is a point of inflection of f and $f'''(c) = 0$. For example, consider $f(x) = x^5$ and $c = 0$.

Example : Let $f(x) = \frac{x^2-4}{x-1} = x + 1 - \frac{3}{x-1}$. Since $f'(x) > 0 \; \forall \; x \neq 1$, the function is increasing on $(-\infty, 1)$ and $(1, \infty)$ and there is no local maximum and no local minimum. Since $f''(x) > 0 \; \forall \; x < 1$ the function is convex on $(-\infty, 1)$ and since $f''(x) < 0 \; \forall \; x > 1$ the function is concave on $(1, \infty)$. There is no point of inflection as $f''(x) \neq 0$ for all $x \neq 1$ and f is not defined at $x = 1$.