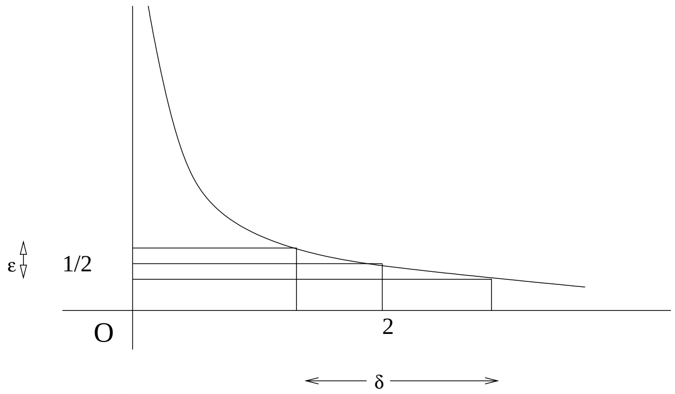
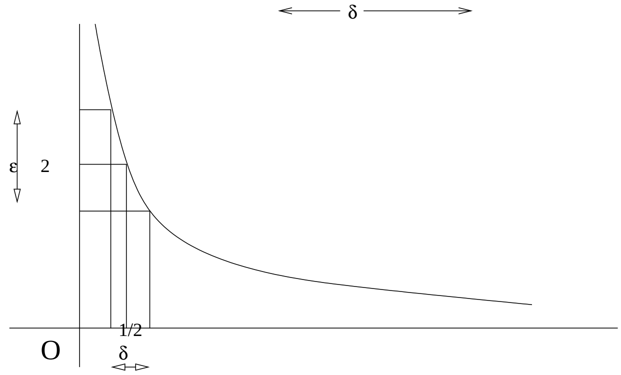
Uniform Continuity

Let us first review the notion of continuity of a function. Let $A \subset I\!R$ and $f : A \to I\!R$ be continuous. Then for each $x_0 \in A$ and for given $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $x \epsilon A$ and $| x - x_0 | < \delta$ imply $| f(x) - f(x_0) | < \varepsilon$. We emphasize that δ depends, in general, on ϵ as well as the point x_0. Intuitively this is clear because the function f may change its values rapidly near certain points and slowly near other points.

For example consider $f(x) = 1/x$. The following two figures explain that for a given ε- neighbourhood about each of $f(2) = 1/2$ and $f(1/2) = 2$, the corresponding maximum values of δ for the points 2 and 1/2 are seen to be different.





We also see that as x_0 tends to 0, the permissible values of δ tends to 0.

Example 1: Consider the function $f : I\!R \to I\!R$ defined by $f(x) = x^2$ for all $x \in I\!R$. Suppose $\varepsilon = 2$ and $x_0 = 1$. Then $f(x) - f(x_0) = x^2 - 1$. If $| x - 1 | < 1/2$ then $1/2 < x < 3/2$ and so $-3/4 < x^2 - 1 < 5/4$. Therefore with $\varepsilon = 2$ and $x_0 = 1$, we have $| x - x_0 | < 1/2$ imply $| f(x) - f(x_0) | < 2$. So $\delta = 1/2$ works in this case.

We will now illustrate that the previous statement is not true for $x_0 = 10$. For, when $x_0 = 10$ we have $f(x) - f(x_0) = x^2 - 100$. If $x = 10 + 1/4$ then $| x - x_0 | < 1/2$ but $f(x) - f(x_0) = (10 + 1/4)^2 - 10^2 > 2$. This shows that even though f is continuous at the point 10 as well at the point 1, for $\varepsilon = 2$ the number $\delta = 1/2$ works for $x_0 = 1$ but not for $x_0 = 10$.

One may ask that for this f, corresponding to $\varepsilon = 2$, there might be some δ (possibly depending on ε) that will work for all $x \in I\!R$. We will show that the answer to this question is negative. Suppose there is a $\delta > 0$ such that for every $x \in I\!R$, we have:

$$| x - y | < \delta \text{ imply } | f(x) - f(y) | < 2.$$

Let $x \in I\!R$ and choose $y = x + \delta/2$. Since $| x - y | < \delta$, by assumption, we have

$$\begin{aligned} | f(x) - f(y) | &= | x - y || x + y | \\ &= \delta/2 \, | 2x + \delta/2 | \\ &= | \delta x + \delta^2/4 | < 2 \end{aligned}$$

This implies that $\delta x < 2$ for all $x \in I\!R^+$, the set of positive real numbers. This is clearly false.

The next example shows that it is not always the case that δ is dependent upon $x_0 \epsilon A$.

Example 2 : Let $A = \{x \in I\!R : x \geq 0\} = [0, \infty) \subset I\!R$ and $f : A \to I\!R$ be defined by

$$f(x) = \sqrt{x}.$$

It is easy to verify that for all $x, y \in A$, $| f(x) - f(y) | = | \sqrt{x} - \sqrt{y} | \leq \sqrt{| x - y |}$. Therefore for given $\varepsilon > 0$ if we choose $\delta = \varepsilon^2$. We have :

$$x, y \in A \text{ and } | x - y | < \delta \text{ imply } | f(x) - f(y) | < \varepsilon.$$

The preceding discussion motivates the following definition.

Definition: A function $f : A \to I\!R$, where $A \subset I\!R$ is said to be *uniformly continuous on* A if given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in A$ and $| x - y | < \delta$, we have $| f(x) - f(y) | < \varepsilon$

Clearly uniform continuity implies continuity but the converse is not always true as seen from Example 1.

In the previous definition we also emphasise that the uniform continuity of f is dependent upon the function f and on the set A. For example, we had seen in Example 1 that the function defined by $f(x) = x^2$ is not uniformly continuous on $I\!R$ or (a, ∞) for all $a \in I\!R$. Let $A = [a, b], a > 0$ and $\varepsilon > 0$. Then

$$| f(x) - f(y) | = | x^2 - y^2 | = | x - y || x + y | \leq 2b \, | x - y |$$

Hence for $\delta = \dfrac{\varepsilon}{2b}$. We have

$$x, y \in I\!R, | x - y | < \delta \Rightarrow | f(x) - f(y) | < \varepsilon$$

Therefore f is uniformly continuous on $[a, b]$.

Infact we illustrate that every continuous function on any closed bounded interval is uniformly continuous.

Let us formulate an equivalent condition to saying that f is not uniformly continuous on A.

Let $A \subset I\!R$ and let $f : A \to I\!R$. Then the following conditions are equivalent.

(i) f is not uniformly condtion on A .

(ii) There exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there are points x, y in A such that $\mid x - y \mid < \delta$ and $\mid f(x) - f(y) \mid \geq \varepsilon_0$.

(iii) There exist an $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A such that $\lim(x_n - y_n) = 0$ and $\mid f(x_n) - f(y_n) \mid \geq \varepsilon_0$ for all $n \in N$.

Example 3: *We can apply this result to show that $g(x) := \dfrac{1}{x}$ is not uniformly continuous on $A := \{x \epsilon I\!R : x > 0\}$. For if $x_n := \dfrac{1}{n}$ and $y_n := \dfrac{1}{n+1}$, then we have $\lim(x_n - y_n) = 0$ but $\mid g(x_n) - g(y_n) \mid = 1$ for all $n \in N$.*

As an immediate consequence of the previous observation, we have the following result which provides us with a sequential criterion for uniform continuity.

Prososition 1: *A function $f : A \to I\!R$ is uniformly continuous on a set $A \subset I\!R$ if and only if whenever sequences (x_n) and (y_n) of points A are such that the sequence $(x_n - y_n)$ converges to 0, the sequence $f(x_n) - f(y_n)$ converges to 0.*

Theorem 2: *Let $a < b$ and $f : [a, b] \to I\!R$ be continuous. Then f is uniformly continuous.*

Proof: Assume the contrary that f is not uniformly continuous. Hence there exist an $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in $[a, b]$ such that $x_n - y_n \to 0$ and $\mid f(x_n) - f(y_n) \mid \geq \varepsilon_0$ for all $n \in N$. Since (x_n) is in $[a, b]$, by Theorem 2.8, there exists a subsequence (x_{ni}) of (x_n) such that $x_{ni} \to x_0 \in [a, b]$. Hence $y_{ni} \to x_0$. By continuity of f, it follows that $f(x_{ni}) \to f(x_0)$ and $f(y_{ni}) \to f(x_0)$. Therefore $\mid f(x_{ni}) - f(y_{ni}) \mid \to 0$. This contradicts the fact that $\mid f(x_{ni}) - f(y_{ni}) \mid \geq \varepsilon_0$. Therefore f is uniformly continuous.

Problems :

1. Let $A \subset I\!R$ and $f : A \to I\!R$ be uniformly continuous on A. Show that if (x_n) is a Cauchy sequence in A then $(f(x_n))$ is a Cauchy sequence in $I\!R$.

2. Using the previous problem show that the following functions are not uniformly continuous.
(i) $f(x) = \dfrac{1}{x^2}, x \in (0, 1)$

(ii) $f(x) = tanx, x \in [0, \dfrac{\pi}{2})$