Practice problems 1: The Real Number System

1. Let \( x_0 \in \mathbb{R} \) and \( x_0 \geq 0 \). If \( x_0 < \epsilon \) for every positive real number \( \epsilon \), show that \( x_0 = 0 \).

2. Prove Bernoulli’s inequality: for \( x > -1 \), \((1 + x)^n \geq 1 + nx \) for all \( n \in \mathbb{N} \).

3. Let \( E \) be a non-empty bounded above subset of \( \mathbb{R} \). If \( \alpha \) and \( \beta \) are supremums of \( E \), show that \( \alpha = \beta \).

4. Suppose that \( \alpha \) and \( \beta \) are any two real numbers satisfying \( \alpha < \beta \). Show that there exists \( n \in \mathbb{N} \) such that \( \alpha < \alpha + \frac{1}{n} < \beta \). Similarly, show that for any two real numbers \( s \) and \( t \) satisfying \( s < t \), there exists \( n \in \mathbb{N} \) such that \( s < t - \frac{1}{n} < t \).

5. Let \( A \) be a non-empty subset of \( \mathbb{R} \) and \( \alpha \in \mathbb{R} \) be an upper bound of \( A \). Suppose for every \( n \in \mathbb{N} \), there exists \( a_n \in A \) such that \( a_n \geq \alpha - \frac{1}{n} \). Show that \( \alpha \) is the supremum of \( A \).

6. Find the supremum and infimum of the set \( \left\{ \frac{-m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z} \right\} \).

7. Let \( E \) be a non-empty bounded above subset of \( \mathbb{R} \). If \( \alpha \in \mathbb{R} \) is an upper bound of \( E \) and \( \alpha \in E \), show that \( \alpha \) is the l.u.b. of \( E \).

8. Let \( x \in \mathbb{R} \). Show that there exists an integer \( m \) such that \( m \leq x < m + 1 \) and an integer \( l \) such that \( x < l \leq x + 1 \).

9. Let \( A \) be a non-empty subset of \( \mathbb{R} \) and \( x \in \mathbb{R} \). Define the distance \( d(x, A) \) between \( x \) and \( A \) by \( d(x, A) = \inf \{|x-a| : a \in A\} \). If \( \alpha \in \mathbb{R} \) is the l.u.b. of \( A \), show that \( d(\alpha, A) = 0 \).

10. (*)

   (a) Let \( x \in \mathbb{Q} \) and \( x > 0 \). If \( x^2 < 2 \), show that there exists \( n \in \mathbb{N} \) such that \( (x + \frac{1}{n})^2 < 2 \). Similarly, if \( x^2 > 2 \), show that there exists \( n \in \mathbb{N} \) such that \( (x - \frac{1}{n})^2 > 2 \).

   (b) Show that the set \( A = \{ r \in \mathbb{Q} : r > 0, r^2 < 2 \} \) is bounded above in \( \mathbb{Q} \) but it does not have the l.u.b. in \( \mathbb{Q} \).

   (c) From (b), conclude that \( \mathbb{Q} \) does not possess the l.u.b. property.

   (d) Let \( A \) be the set defined in (b) and \( \alpha \in \mathbb{R} \) such that \( \alpha = \sup A \). Show that \( \alpha^2 = 2 \).

11. (*) For a subset \( A \) of \( \mathbb{R} \), define \( -A = \{ -x : x \in A \} \). Suppose that \( S \) is a nonempty bounded subset of \( \mathbb{R} \).

   (a) Show that \( -S \) is bounded below.

   (b) Show that \( \inf(-S) = -\sup(S) \).

   (c) From (b) conclude that the l.u.b. property of \( \mathbb{R} \) implies the g.l.b. property of \( \mathbb{R} \) and vice versa.

12. (*) Let \( k \) be a positive integer and \( x = \sqrt{k} \). Suppose \( x \) is rational and \( x = \frac{m}{n} \) such that \( m \in \mathbb{Z} \) and \( n \) is the least positive integer such that \( nx \) is an integer. Define \( n' = n(x - [x]) \) where \( [x] \) is the integer part of \( x \).

   (a) Show that \( 0 \leq n' < n \) and \( n'x \) is an integer.

   (b) Show that \( n' = 0 \).

   (c) From (a) and (b) conclude that \( \sqrt{k} \) is either a positive integer or irrational.
Hints/Solutions

1. Suppose \( x_0 \neq 0 \). Then for \( \epsilon_0 = \frac{\text{49}}{2} \), \( x_0 > \epsilon_0 > 0 \) which is a contradiction.

2. Use Mathematical induction.

3. Since \( \alpha \) is a l.u.b. of \( E \) and \( \beta \) is an u.b. of \( E \), \( \alpha \leq \beta \). Similarly \( \beta \leq \alpha \).

4. Since \( \beta - \alpha > 0 \), by Archimedean property, there exists \( n \in \mathbb{N} \) such that \( n > \frac{1}{\beta - \alpha} \).

5. If \( \alpha \) is not the l.u.b then there exists an u.b. \( \beta \) of \( A \) such that \( \beta < \alpha \). Find \( n \in \mathbb{N} \) such that \( \beta < \alpha - \frac{1}{n} \). Since \( \exists a_n \in A \) such that \( \alpha - \frac{1}{n} < a_n \), \( \beta \) is not an u.b. which is a contradiction.

6. \( \sup = 1 \) and \( \inf = -1 \).

7. If \( \alpha \) is not the l.u.b. of \( E \), then there exists an u.b. \( \beta \) of \( E \) such that \( \beta < \alpha \). But \( \alpha \in E \) which contradicts the fact that \( \beta \) is an u.b. of \( E \).

8. Using the Archimedean property, find \( m, n \in \mathbb{N} \) such that \(-m < x < n \). Let \( [x] \) be the largest integer between \(-m \) and \( n \) such that \( [x] \leq x \). So, \( [x] \leq x < [x] + 1 \). This implies that \( x < [x] + 1 \). Take \( l = [x] + 1 \). \( ([x] \) is called the integer part of \( x \).

9. If \( d(\alpha, A) > 0 \), then find \( \epsilon \in \mathbb{R} \) such that \( 0 < \epsilon < d(\alpha, A) \). So \( \alpha - a > \epsilon \) for all \( a \in A \). That is \( a < \alpha - \epsilon \) for all \( a \in A \). Hence \( \alpha - \epsilon \) is an u.b. of \( A \) which is contradiction.

10. (a) Suppose \( x^2 < 2 \). Observe that \( (x + \frac{1}{n})^2 < x^2 + 2x + \frac{2}{n} \) for any \( n \in \mathbb{N} \). Using the Archimedean property, find \( n \) such that \( x^2 + \frac{1}{n} + \frac{2}{n} < 2 \). This \( n \) will do.

   (b) Note that 2 is an u.b. of \( A \). If \( m \in \mathbb{Q} \) such that \( m = \sup A \), then there are three possibilities: i. \( m^2 < 2 \) ii. \( m^2 = 2 \) iii. \( m^2 > 2 \). Using (a) show that this is not possible.

   (c) The set \( A \) defined in (b) is bounded above in \( \mathbb{Q} \) but does not have the l.u.b. in \( \mathbb{Q} \).

   (d) Using (a), justify that the following cases cannot occur: (i) \( \alpha^2 < 2 \) and (ii) \( \alpha^2 > 2 \).

11. (a) Trivial.

   (b) Let \( \alpha = \sup S \). We claim that \(-\alpha = \inf(-S) \). Since \( \alpha = \sup S \), \( a \leq \alpha \) for all \( a \in S \). This implies that \( -a \geq -\alpha \) for all \( a \in S \). Hence \(-\alpha \) is a l.b. of \(-S \). If \(-\alpha \) is not the g.l.b. of \(-S \) then there exists a lower bound \( \beta \) of \( A \) such that \(-\alpha < \beta \). Verify that \(-\beta \) is an u.b. of \( S \) and \(-\beta < \alpha \) which is a contradiction.

   (c) Assume that \( \mathbb{R} \) has the l.u.b. property and \( S \) is a non empty bounded below set. Then from (b) or the proof of (b), we conclude that \( \inf S \) exists and is equal to \(-\sup(-S) \).

12. Trivial.