Practice Problems 12 : Comparison, Limit comparison and Cauchy condensation tests

1. Let $a_n \geq 0$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} a_n$ converges then show that

   (a) $\sum_{n=1}^{\infty} a_n^2$ converges. Is the converse true ?

   (b) $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges.

   (c) $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

   (d) $\sum_{n=1}^{\infty} \frac{a_n+4^n}{a_n+5^n}$ converges using comparison or limit comparison test.

2. Let (a_n) be a sequence such that $a_n > 0$ for all n and $a_n \to \infty$. Show that $\sum_{n=1}^{\infty} \frac{1}{a_n^n}$ converges.

3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that $\sum_{n=1}^{\infty} |a_n|$ diverges if $\sum_{n=1}^{\infty} a_n^2$ diverges.

4. Let $a_n > 0$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} \frac{a_1+a_2+\ \dots\ +a_n}{n}$ diverges.

5. Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ converges. Does the converse hold ?

6. Let $a_n, b_n \in \mathbb{R}$ for all n and $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Show that $\sum_{n=1}^{\infty}(a_n - b_n)^p$ converges for all $p \geq 2$.

7. Show that $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{n+1}\right)$ diverges.

8. Let $a_n \geq 0$ for all n and $n^3 a_n^2 \to \ell$ for some $\ell > 0$. Show that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ converges.

9. Suppose $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Show that the series $\sum_{n=1}^{\infty}\left(1 - \frac{\sin a_n}{a_n}\right)$ converges.

10. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n}$ for $n = 1, 4, 9, 16, \dots$ and $a_n = \frac{1}{n^2}$ otherwise (i.e., if n is not a perfect square). Show that $\sum_{n=1}^{\infty} a_n$ converges but $na_n \nrightarrow 0$.

11. Let (a_n) be a sequence of positive real numbers such that $a_{n+1} \leq a_n$ for all n and $\sum_{n=1}^{\infty} a_n$ converge. Show that $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ converges.

12. Show that $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$ diverges.

13. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=2}^{\infty} a_n$ where a_n equals:

    (a) $\frac{1}{(\ln n)^p}, (p > 0)$ (b) $\frac{\sin(\frac{1}{n})}{\sqrt{n}}$ (c) $\frac{\ln n}{\sqrt{n}}$ (d) $\frac{1}{n^2 - \ln n}$ (e) e^{-n^2}

    (f) $\frac{1}{n^{1+\frac{1}{n}}}$ (g) $\tan\frac{1}{n}$ (h) $1 - \cos\frac{\pi}{n}$ (i) $(\ln n)\sin\frac{1}{n^2}$ (j) $\frac{\tan^{-1} n}{n\sqrt{n}}$

    (k) $(n+2)(1 - \cos\frac{1}{n})$ (ℓ) $\frac{3+\cos n}{e^n}$ (m) $\frac{2+\sin^3(n+1)}{2^n+n^2}$ (n) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$

14. (*) Suppose that $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ diverges. Let (A_n) be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and (S_n) be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{a_n}{A_n}$

    (a) Show that (S_n) does not satisfy the Cauchy criterion.

    (b) Show that there exists a sequence (b_n) such that $b_{n+1} \leq b_n$ for all n, $b_n \to 0$ and $\sum_{n=1}^{\infty} b_n a_n$ also diverges.

Practice Problems 12 : Hints/Solutions

1. (a) Since $a_n \to 0$, $a_n^2 \le a_n$ eventually. Converse is not true: Take $a_n = n^{-\frac{2}{3}}$.

   (b) Use the inequality $\sqrt{a_n a_{n+1}} \le \frac{1}{2}(a_n + a_{n+1})$.

   (c) Use $\sqrt{a_n \frac{1}{n^2}} \le \frac{1}{2}(a_n + \frac{1}{n^2})$.

   (d) Use $\frac{a_n+4^n}{a_n+5^n} \le \frac{a_n+4^n}{5^n} \le \left(\frac{1}{5}\right)^n + \left(\frac{4}{5}\right)^n$ or apply LCT with $\left(\frac{4}{5}\right)^n$, i.e., find $\lim_{n\to\infty} \frac{a_n+4^n}{a_n+5^n}\left(\frac{5}{4}\right)^n$.

2. Observe that $\frac{1}{a_n^n} < \frac{1}{2^n}$ eventually.

3. Since $a_n \to 0$, $a_n^2 \le |a_n|$ eventually.

4. Note that $\frac{a_1+a_2+\ \dots\ +a_n}{n} \ge \frac{a_1}{n}$.

5. Use the inequality $a_n^2 + b_n^2 \le (a_n + b_n)^2$. Converse is true because $a_n \le \sqrt{a_n^2 + b_n^2}$.

6. It is sufficient to show that $\sum_{n=1}^{\infty}(a_n - b_n)^2$ converges because $|a_n - b_n|^p \le (a_n - b_n)^2$ eventually for $p > 2$. For convergence of $\sum_{n=1}^{\infty}(a_n - b_n)^2$, use the inequality $(a - b)^2 = 2a^2 + 2b^2 - (a + b)^2 \le 2a^2 + 2b^2$.

7. Use the LCT with $\frac{1}{n}$: $n \sin\left(\frac{n\pi}{n+1}\right) \to \pi$.

8. Use the LCT with $\frac{1}{n^2}$: $\frac{a_n}{\sqrt{n}}\frac{n^2}{1} = a_n n^{\frac{3}{2}} \to \sqrt{\ell} > 0$.

9. Use the LCT with a_n^2: $\frac{1}{a_n^2}\left(1 - \frac{\sin a_n}{a_n}\right) = \frac{a_n - \sin a_n}{a_n^3} \to \frac{1}{6}$.

10. The series is $\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9} + \dots$ The sequence of partial sums is bounded above by $(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots) + (1 + \frac{1}{4} + \frac{1}{9} + \dots) \le 2\sum_{n=1}^{\infty}\frac{1}{n^2}$ but $na_n = 1$ when n is a perfect square.

11. The partial sum S_n of $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ is $a_1 + a_2 + \dots + a_n - na_{n+1}$.

12. Use the Cauchy condensation test and the fact that $\ln 2 < 1$.

13. (a) Diverges (Use the LCT with $\frac{1}{n}$: $\frac{n}{(\ln n)^p} \to \infty$).

    (b) Converges (Use the LCT with $\frac{1}{n\sqrt{n}}$).

    (c) Diverges (Use the LCT with $\frac{1}{\sqrt{n}}$).

    (d) Converges (Use the comparison test: $\frac{1}{n^2 - \ln n} \le \frac{1}{n^2 - n} \le \frac{1}{n(n-1)}$).

    (e) Converges (Use the comparison test: $\frac{1}{e^{n^2}} \le \frac{1}{n^2}$ as $e^x \ge x$).

    (f) Diverges (Use the LCT with $\frac{1}{n}$: $\frac{n}{n^{1+\frac{1}{n}}} \to 1$).

    (g) Diverges (Use the LCT with $\frac{1}{n}$: $\lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sec^2(\frac{1}{n})(-\frac{1}{n^2})}{-\frac{1}{n^2}} = 1$).

    (h) Converges (Use the LCT with $\frac{1}{n^2}$: $\frac{1-\cos\frac{\pi}{n}}{\frac{1}{n^2}} \to \frac{\pi^2}{2}$).

    (i) Converges (Use the LCT with $\frac{1}{n\sqrt{n}}$: $\frac{(\ln n)\sin\frac{1}{n^2}}{\frac{1}{n\sqrt{n}}} = \frac{\ln n}{\sqrt{n}}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}$).

    (j) Converges (Use the comparison test: $\frac{\tan^{-1}}{n\sqrt{n}} \le \frac{\frac{\pi}{2}}{n\sqrt{n}}$).

(k) Diverges because $(n+2)(1-\cos\frac{1}{n}) \geq n(1-\cos\frac{1}{n})$ and $\sum_{n=1}^{\infty} n(1-\cos\frac{1}{n})$ diverges: $\frac{n(1-\cos\frac{1}{n})}{\frac{1}{n}} = \frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}} \to \frac{1}{2}$.

(ℓ) Converges (Use the comparison test: $0 \leq \frac{3+\cos n}{e^n} \leq \frac{4}{e^n} = 4(\frac{1}{e})^n$).

(m) Converges because both $\sum_{n=1}^{\infty} \frac{2}{2^n+n^2}$ and $\sum_{n=1}^{\infty} \left|\frac{\sin^3(n+1)}{2^n+n^2}\right|$ converge.

(n) Converges because $\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n}\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{n^{\frac{3}{2}}}$.

14. (a) Note that, for any $p \in \mathbb{N}$, $|S_{n+p} - S_n| \geq \frac{a_{n+1}+a_{n+2}+\ \dots\ +a_{n+p}}{A_{n+p}} = \frac{A_n+p-A_n}{A_{n+p}} \to 1$ as $p \to \infty$.

(b) Take $b_n = \frac{1}{A_n}$.