1. For a given \( \sum_{n=0}^{\infty} a_n x^n \), let \( K = \left\{ |x| : x \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} a_n x^n \text{ is convergent} \right\} \) be bounded. If \( r = \sup K \), then \( \sum_{n=0}^{\infty} a_n x^n \)

(a) converges absolutely for all \( x \in \mathbb{R} \) with \( |x| < r \),

(b) diverges for all \( x \in \mathbb{R} \) with \( |x| > r \).

2. In each of the following cases, determine the values of \( x \) for which the power series converges.

(a) \( \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \)

(b) \( \sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!} \)

(c) \( \sum_{n=0}^{\infty} (-1)^n n^2 x^n \)

(d) \( \sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{n^3} \)

(e) \( \sum_{n=0}^{\infty} (-1)^n \frac{10^n}{n!} (x - 10)^n \)

3. Determine the values of \( x \) for which the series \( \sum_{n=0}^{\infty} \frac{x^n (\ln n)^2}{n^2} \) converges absolutely.

4. Let \( (S_n) \) be the sequence of partial sums of the Maclaurin series of \( \ln(1 + x) \). Show that if \( 0 \leq x \leq 1 \), then \( S_n \to \ln(1 + x) \), i.e, the Maclaurin series of \( \ln(1 + x) \) converges to \( \ln(1 + x) \) on \([0, 1]\).

5. Let \( f : (a, b) \to \mathbb{R} \) be infinitely differentiable and \( x_0 \in (a, b) \). Suppose that there exists \( M > 0 \) such that \( |f^{(n)}(x)| \leq M^n \) for all \( n \in \mathbb{N} \) and \( x \in (a, b) \). Show that Taylor’s series of \( f \) around \( x_0 \) converges to \( f(x) \) for all \( x \in (a, b) \).

6. Estimate the upper bound on the error if we consider \( P_2(x) = 1 + x + x^2 \) as an approximation for \( e^x \) on \([0, 1]\).

7. Let \( f(x) = e^{-\frac{1}{x^2}} \) when \( x \neq 0 \) and \( f(0) = 0 \). Show that

(a) \( f'(0) = 0 \).

(b) for \( x \neq 0 \), \( n \geq 1 \), \( f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x^2}} \) where \( P_n \) is a polynomial of degree \( 3n \).

(c) \( f^{(n)}(0) = 0 \) for \( n = 1, 2, \ldots \).

(d) the Maclaurin series of \( f \) converges to \( f(x) \) only when \( x = 0 \).

8. (*) Let \( a_n \geq 0 \) for all \( n \in \mathbb{N} \) and \( (a_n) \) be a bounded sequence. For each \( n \), define

\[ A_n = \sup \{ a_k^{\frac{1}{k}} : k \geq n \} \]

(see Problem 12 in Practice Problems 2). Since \( (A_n) \) converges, let \( A_n \to \ell \) for some \( \ell > 0 \).

(a) If \( \ell < 1 \), the series \( \sum_{n=1}^{\infty} a_n \) converges and if \( \ell > 1 \), the series diverges.

(b) The radius of convergence of the power series \( \sum_{n=1}^{\infty} a_n x^n \) is \( \frac{1}{\ell} \)

(c) Find the radius of convergence of the power series

\[ \frac{1}{2} x + \frac{1}{3} x^3 + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots \]
1. (a) If $|x| < r$, then by the definition of supremum there exists $|x_0| \in K$ such that $|x| < |x_0|$. Since $\sum_{n=0}^{\infty} a_n x^n$ converges, by Theorem 1, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

(b) Suppose $|x| > r$. By the definition of $K$, $\sum_{n=0}^{\infty} a_n x^n$ diverges.

2. (a) Since $\left| \frac{x^{n+1}}{n+1} \right| \to 0$, by the root test the series converges for all $x \in \mathbb{R}$.

(b) In this case $\left| \frac{a_n x^{n+1}}{a_n x^n} \right| \to |\frac{1}{x}|$ and $\frac{a_n+1}{x^{n+1}} = \frac{(n+1)}{(n+2)} > 1$. The series converges only for $|x| < 4$ as $(a_n 4^n)$ increases and $a_n 4^n \to 0$.

(c) Use Ratio test. The series converges only for $|x| < \frac{1}{2}$.

(d) Use Ratio test. The series converges for $|x - 2| < 3$, and hence for $-1 < x < 5$. At $x = 5$ the series diverges and $x = -1$ the series diverges.

(e) Since $\frac{a_{n+1}}{a_n} (x - 10) \to 0$, the series converges for all $x \in \mathbb{R}$.

3. Apply the Ratio test. The series converges absolutely if and only if $x \in [-1, 1]$.

4. By Taylor’s theorem $\ln(1 + x) = S_n + \frac{(-1)^n x^{n+1}}{n+1} \frac{(1+c)^{n+1}}{n+1}$ for some $c \in (0, x)$. This implies that $\left| \ln(1 + x) - S_n \right| = \frac{(-1)^n x^{n+1}}{n+1} \frac{(1+c)^{n+1}}{n+1} \leq \frac{|x|^{n+1}}{n+1} \to 0$.

5. Note that for $x \in (a, b)$, $|E_n(x)| = \left| \frac{(n+1)c}{(n+1)!} \right| |x - x_0|^{n+1}$ for some $c$ between $x$ and $x_0$. This implies that $|E_n(x)| \leq \frac{A}{(n+1)!}$ where $A = M|x - x_0|$. It follows from the ratio test for sequences that $\frac{A}{(n+1)!} \to 0$. This shows that Taylor’s series of $f$ converges to $f(x)$.

6. Note that $|E_2(x)| = |f(x) - P_2(x)| \leq \frac{x^4}{6} |x|^3 \leq \frac{0.01 \times 0.001}{6}$.

7. (a) Note that $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{\frac{1}{e^{x^2}}}{x} = \lim_{x \to 0^+} \frac{\frac{1}{e^{x^2}}}{x} = \lim_{y \to \infty} \frac{y}{e^{y^2}} = 0$, by L’Hospital Rule.

(b) If $f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x^2}}$, then
$$f^{(n+1)}(x) = \left\{ P_n'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + P_n\left(\frac{1}{x}\right)\left(\frac{2}{x^3}\right) \right\} e^{-\frac{1}{x^2}} = P_{n+1}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}$$

where $P_{n+1}(t) = -t^2 P_n'(t) + 2t^3 P_n(t)$ which is of degree $3n + 3$ if $P_n$ is of degree $3n$. Use induction argument.

(c) If $f^{(n-1)}(0) = 0$ then, as done in (a), $\lim_{x \to 0^+} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x} = \lim_{y \to \infty} \frac{y P_{n-1}(y)}{e^{y^2}} = 0$, i.e., $f^n(0) = 0$.

(d) Trivial.

8. (a) If $\ell < 1$, then find $\epsilon > 0$ such that $\ell < \ell + \epsilon < 1$. Since $A_n \to \ell$, there exists $N \in \mathbb{N}$ such that $A_n < \ell + \epsilon$ for all $n \geq N$. That is $\frac{1}{n} < \ell + \epsilon < 1$ for all $n \geq N$. Therefore by the Root Test the series $\sum_{n=1}^{\infty} a_n$ converges.

If $\ell > 1$, choose $\epsilon > 0$ such that $\ell - \epsilon > 1$. Since $A_n \to \ell$, there exists a subsequence $(a_{n_k})$ of $(a_n)$ such that $a_{n_k} \geq \ell - \epsilon > 1$. Hence $a_{n_k} \to 0$ and therefore $\sum_{n=1}^{\infty} a_n$ diverges.

(b) Follows from the proof of (a) (Repeat the proof of (a) by replacing $a_n$ by $a_n x^n$).

(c) See Problem 5 of Practice Problems 13. In this case $\ell = \frac{1}{\sqrt{2}}$ and hence $\frac{1}{\ell} = \sqrt{2}$ is the radius of convergence.