1. Let \( f : [a, b] \to \mathbb{R} \) be integrable and \([c, d] \subset [a, b] \). Show that \( f \) is integrable on \([c, d] \).

2. (a) Let \( f \) be bounded on \([c, d] \), \( M = \sup\{ f(x) : x \in [c, d] \} \), \( M' = \sup\{ |f(x)| : x \in [c, d] \} \), \( m = \inf\{ f(x) : x \in [c, d] \} \) and \( m' = \inf\{ |f(x)| : x \in [c, d] \} \). Show that \( M' - m' \leq M - m \).
   
   (b) Let \( f : [a, b] \to \mathbb{R} \) be integrable. Show that \( |f| \) and \( f^2 \) are integrable.

3. (a) Let \( f : [0, 1] \to \mathbb{R} \) be integrable. Show that \( f \) is integrable on \([0, 1] \).
   
   (b) Let \( f : [0, 1] \to \mathbb{R} \) such that \( f^2 \) is integrable but \( f \) is not integrable. Find \( f \).

4. Let \( f \) and \( g \) be two integrable functions on \([a, b] \).
   
   (a) If \( f(x) \leq g(x) \) for all \( x \in [a, b] \), show that \( \int_a^b f(x)dx \leq \int_a^b g(x)dx \).
   
   (b) Show that \( |\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx \).
   
   (c) If \( m \leq f(x) \leq M \) for all \( x \in [a, b] \), show that \( m(b - a) \leq \int_a^b f(x)dx \leq M(b - a) \). Use this inequality to show that \( \int_0^{\pi/3} x dx \leq \frac{\sqrt{3}}{6} \).

5. Let \( f : [a, b] \to \mathbb{R} \) and \( f(x) \geq 0 \) for all \( x \in [a, b] \).
   
   (a) If \( f \) is integrable, show that \( \int_a^b f(x)dx \geq 0 \).
   
   (b) If \( f \) continuous and \( \int_a^b f(x)dx = 0 \) show that \( f(x) = 0 \) for all \( x \in [a, b] \).
   
   (c) Give an example of an integrable function \( f \) on \([a, b] \) such that \( f(x) \geq 0 \) for all \( x \in [a, b] \) and \( \int_a^b f(x)dx = 0 \) but \( f(x_0) \neq 0 \) for some \( x_0 \in [a, b] \).

6. Let \( f : [0, 1] \to \mathbb{R} \) be a bounded function. Suppose that for any \( c \in (0, 1] \), \( f \) is integrable on \([c, 1] \).
   
   (a) Show that \( f \) is integrable on \([0, 1] \).
   
   (b) Show that the function \( f \) defined by \( f(0) = 0 \) and \( f(x) = \sin(\frac{1}{x}) \) on \([0, 1] \) is integrable,

7. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Suppose that whenever the product \( fg \) is integrable on \([a, b] \) for some integrable function \( g \), we have \( \int_a^b (fg)(x)dx = 0 \). Show that \( f(x) = 0 \) for every \( x \in [a, b] \).

8. (a) Let \( x, y \geq 0 \). Show that \( \lim_{n \to \infty} (x^n + y^n)^{\frac{1}{n}} = M \) where \( M = \max\{x, y\} \).
   
   (b) Let \( f : [a, b] \to \mathbb{R} \) be continuous and \( f(x) \geq 0 \) for all \( x \in [a, b] \). Show that \( \lim_{n \to \infty} \left( \int_a^b f(x)^n \right)^{\frac{1}{n}} = M \) where \( M = \sup\{ f(x) : x \in [a, b] \} \).

9. (a) (Cauchy-Schwarz inequality) Let \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathbb{R} \). By observing that \( \sum_{i=1}^n (x_i + y_i)^2 \geq 0 \) for any \( t \in \mathbb{R} \), show that \( \left| \sum_{i=1}^n x_i y_i \right| \leq \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n y_i^2 \right)^{\frac{1}{2}} \).
   
   (b) (Cauchy-Schwarz inequality) Let \( f \) and \( g \) be any two integrable functions on \([a, b] \).
   
   Show that \( \left( \int_a^b f(x)g(x)dx \right)^2 \leq \left( \int_a^b |f(x)|^2 dx \right) \left( \int_a^b |g(x)|^2 dx \right) \).

10. (*) Let \( f : [a, b] \to \mathbb{R} \) be integrable. Suppose that the values of \( f \) are changed at a finite number of points. Show that the modified function is integrable.

11. (*) Let \( f : [a, b] \to \mathbb{R} \) be a bounded function and \( E \subset [a, b] \). Suppose that \( E \) can be covered by a finite number of closed intervals whose total length can be made as small as desired. If \( f \) is continuous at every point outside \( E \), show that \( f \) is integrable.
1. Let $\epsilon > 0$. Since $f$ is integrable on $[a, b]$, there exists a partition $P = \{x_0, x_1, x_2, ..., x_n\}$ (of $[a, b]$) such that $U(P, f) - L(P, f) < \epsilon$. Let $P_1 = P \cup \{c, d\}$ and $P' = P_1 \cap [c, d]$ which is a partition of $[c, d]$. Then, since $m_i - m_i > 0$, it follows that $U(P', f) - L(P', f) \leq U(P_1, f) - L(P_1, f) < \epsilon$. Apply the Riemann Criterion.

2. (a) Let $x, y \in [c, d]$. Then $|f(x) - f(y)| \leq |f(x) - f(y)| \leq M - m$. Fix $y$ and take supremum for $x$, we get $M - |f(y)| \leq M - m$. Take infimum for $y$.

(b) To show that $|f|$ is integrable, use the Riemann Criterion and (a).

For showing $f^2$ is integrable, use the inequality $(f(x))^2 - (f(y))^2 \leq 2K|f(x) - f(y)|$ where $K = \sup\{|f(x)| : x \in [a, b]\}$ and proceed as in (a).

3. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = -1$ for $x$ rational and $f(x) = 1$ for $x$ irrational. Then $|f| = f^2$. Note that $f$ is not integrable but $|f|$ is a constant function.

4. (a) Use $\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g - f)(x)dx$ and Problem 3 of Practice Problems 15.

(b) Since $-|f(x)| \leq f(x) \leq |f(x)|$, $x \in [a, b]$, (b) follows from part (a).

(c) Use part (a) or $L(P, f) \leq \int_a^b f(x)dx \leq U(P, f)$. On $[\frac{\pi}{4}, \frac{\pi}{3}]$, $\sin \frac{x}{x}$ decreases.

5. (a) This follows from the definition of integrability of $f$ or from Problem 4.

(b) Let $x_0 \in (a, b)$ be such that $f(x_0) > \alpha$ for some $\alpha > 0$. Then by the continuity of $f$ there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and $f(x) > \alpha$ on $(x_0 - \delta, x_0 + \delta)$. Then we can find a partition $P$ of $[a, b]$ such that $J_1 f(x)dx \geq L(P, f) > \alpha \times \delta > 0$.

(c) Let $f(a) = 1$ and $f(x) = 0$ for all $x \in [a, b]$. Then $\int_a^b f(x)dx = 0$ but $f(a) \neq 0$.

6. (a) Let $M = \sup\{|f(x)| : x \in [0, 1]\}$. If $P_n = \{\frac{1}{n}, x_1, x_2, ..., x_n\}$ is a partition of $[\frac{1}{n}, 1]$ then let $P_n' = \{0, \frac{1}{n}, x_1, x_2, ..., x_n\}$ be a corresponding partition of $[0, 1]$. Then $U(P_n', f) \leq \frac{M}{n} + U(P_n, f)$ and $L(P_n', f) \geq -\frac{M}{n} + L(P_n, f)$. Therefore, $U(P_n', f) - L(P_n', f) \leq \frac{2M}{n} + U(P_n, f) - L(P_n, f)$. For $\epsilon > 0$, first choose $n$ such that $\frac{2M}{n} < \frac{\epsilon}{2}$ and then choose $P_n$ such that $U(P_n, f) - L(P_n, f) < \frac{\epsilon}{2}$. Apply the Riemann Criterion.

(b) Since $f$ is continuous on $[c, 1]$ for every $c$ satisfying $0 < c < 1$, $f$ is integrable on $[c, 1]$. Apply part (a).

7. Suppose $f(x_0) > 0$ for some $x_0 \in (a, b)$. Use the argument used in Problem 5(b).

8. (a) Note that $M \leq (x^n + y^n)^\frac{1}{n} \leq (2M^n)^\frac{1}{n}$. Use the Sandwich Theorem.

(b) For $\epsilon > 0$, by the continuity of $f$, $\exists [c, d] \subset [a, b]$ such that $f(x) > M - \epsilon \forall x \in [c, d]$. Hence $(M - \epsilon)(d - c)^\frac{1}{n} \leq \left(\int_a^b f(x)^n dx\right)^\frac{1}{n} \leq M(b - a)^\frac{1}{n}$. Apply the Sandwich Theorem.

9. We will see the solution of part (b) and the solution of part (a) is similar. Note that the inequality $\int_a^b t f(x) - g(x) dx = t^2 \left(\int_a^b f^2(x) dx\right) - 2t \left(\int_a^b f(x)g(x) dx\right) + \int_a^b (g^2(x) dx) \geq 0$ holds for all $t \in \mathbb{R}$. Take $t = \frac{\alpha}{\beta}$ where $\alpha = \int_a^b f(x)g(x) dx$ and $\beta = \int_a^b f^2(x) dx$.

10. Suppose the values of $f$ are changed at $c_1, c_2, ..., c_p$ and $g$ is the modified function. Let $M = \max\{|g(c_1)|, |g(c_2)|, ..., |g(c_p)|\}$. Let $\epsilon > 0$. Since $f$ is integrable, there exists a partition $P$ of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{2}$. Cover $c_i$’s by the intervals $[y_1, y_2], \ [y_3, y_4], ..., [y_{2p-1}, y_{2p}]$ where $y_i$’s are in $[a, b]$ and $|y_1 - y_2| + |y_3 - y_4| + ... + |y_{2p-1} - y_{2p}| < \frac{\epsilon}{4M}$. Consider the partition $P_1 = P \cup \{y_1, y_2, ..., y_{2p}\}$. Then $U(P_1, g) - L(P_1, g) \leq U(P_1, f) - L(P_1, f) + \frac{2\epsilon M}{4M} < U(P, f) - L(P, f) + \frac{\epsilon}{2} \leq \epsilon$. Apply the Riemann Criterion.