Practice Problems 21 : Washer and Shell methods, Length of a plane curve

1. Find the volume of the solid generated by revolving the region bounded by the the curves $y = x^2$ and $x = y^2$ about the y-axis.

2. Let S denote the solid hemisphere $x^2+y^2+z^2 \leq 4, y \geq 0$ and C denote the cone generated by revolving the line $\sqrt{3}y = x$ around the y-axis. Find the volume of the portion of S that lies inside C.

3. Consider the region R in the plane bounded by $y = \sin x, y = 0$ and $x = \frac{\pi}{2}$. Using washer method, find the volume of the solid generated by revolving R about the y-axis.

4. Let R be the region bounded by $y = 6\cos x,\ y = e^x,\ x = 0$ and $x = \frac{\pi}{6}$. Using washer method, evaluate the volume of the solid generated by revolving R around the line $y = 7$

5. Let R be the region enclosed by $y = e^{x^2},\ x = 1, x = 0$ and $y = 0$. The region R is revolved about the y-axis. Find the volume of the solid generated.

6. Find the volume of the solid generated by revolving the region bounded by $(y-2)^2 = 4-x$ and $x = 0$ about the x-axis.

7. A cylindrical hole of radius $\sqrt{3}$ is drilled through the center of the solid sphere of radius 2. Compute the volume of the remaining solid using the Shell Method.

8. Let R be the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$. Find the volume of the solid generated by revolving R about the line $x = 7$ using
   (a) the Washer Method
   (b) the Shell Method.

9. Let C denote the circular disc of radius b centered at $(a, 0)$ where $0 < b < a$. Find the volume of the torus that is generated by revolving C around the y-axis using
   (a) the Washer Method
   (b) the Shell Method.

10. Find the lengths of the following curves.
    (a) $y = \frac{2}{3}(x^2+1)^{\frac{3}{2}},\ x \in [1, 5]$
    (b) $x(t) = 3\sin(2t) - 6t$ and $y(t) = 6\sin^2 t,\ \ 0 \leq t \leq \frac{\pi}{2}$
    (c) $r = \sin^2(\frac{\theta}{2}),\ \ 0 \leq \theta \leq \pi$.

11. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable and increasing function such that $f(0) = 1$. Let $s(x)$ denote the length of the curve $y = f(x)$ from the point $(0, 1)$ to $(x, f(x)),\ x > 0$. Suppose $s(x) = 2x$ for all $x \in [0, \infty)$. Evaluate $f(x)$.

12. Consider the curve $r = e^{-\theta},\ \theta \in [0, \infty)$. Sketch the curve and show that $\int_0^\infty \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta = \sqrt{2}$.

13. Consider the curve $r = \frac{1}{1+\theta},\ \theta \in [0, \infty)$. Sketch the curve and show that $\int_0^\infty \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta$ does not exist.

Practice Problems 21 : Hints/Solutions

1. Solving $y^2 = \sqrt{y}$ implies that $y = 0$ or $y = 1$. The required volume is $\int_0^1 \pi \left[(\sqrt{y})^2 - (y^2)^2\right] dy$. See Figure 1.

2. The volume of the portion of S that lies outside C, evaluated by the Washer Method, is $\int_0^1 \pi(4 - y^2 - 3y^2)dy = \frac{8\pi}{3}$. The required volume is $\frac{16\pi}{3} - \frac{8\pi}{3}$. See Figure 2.

3. The required volume $V = V_1 - V_2$ where $V_1 = \pi \int_0^1 (\frac{\pi}{2})^2 dy$ and $V_2 = \pi \int_0^1 (\sin^{-1} y)^2 dy$. The substitution $t = \sin^{-1} y$ gives that $V_2 = \pi \int_0^{\frac{\pi}{2}} t^2 \cos t dt$ which can be evaluated using integration by parts. See Figure 3.

4. For $x \in [0, \frac{\pi}{6}]$, $7 > 6\cos x \geq 6 \cos\frac{\pi}{6} = 6\frac{\sqrt{3}}{2} > e > e^{\frac{\pi}{6}} \geq e^x$. Therefore the required volume is $\int_0^{\frac{\pi}{6}} \pi \left[(7 - e^x)^2 - (7 - 6\cos x)^2\right] dx$. See Figure 4.

5. By the Shell Method, the required volume is $\int_0^1 2\pi x e^{x^2} dx = \pi \int_0^1 e^u du$.

6. The graph intersects the y -axis at $(0, 0)$ and $(0, 4)$. The volume, determined by the Shell Method, is $\int_0^4 2\pi y(4 - (y - 2)^2)dy$. See Figure 5.

7. The required volume, determined by the Shell Method, is $\int_{\sqrt{3}}^2 2\pi x 2y dx = 4\pi \int_{\sqrt{3}}^2 x\sqrt{4 - x^2}dx$ $= \frac{4\pi}{3}$. See Figure 6.

8. (a) See Figure 7. The volume is $\pi \int_0^4 \left\{ \left[7 - (\frac{y^2}{4} + 1)\right]^2 - [7 - (y + 1)]^2 \right\} dy$.

   (b) See Figure 8. The volume is $\int_1^5 2\pi(7 - x) \left[(2\sqrt{x - 1} - (x - 1)\right] dx$.

9. (a) See Figure 9. Note that the disc is bounded by the curves $x = a + \sqrt{b^2 - y^2}$ and $x = a - \sqrt{b^2 - y^2}$. The volume of the torus, evaluated by the Washer Method, is $\pi \int_{-b}^b \left((a + \sqrt{b^2 - y^2})^2 - (a - \sqrt{b^2 - y^2})^2\right) dy = 4a\pi \int_{-b}^b \sqrt{b^2 - y^2}dy$. The last integral is the area of the semicircle of radius b. Therefore the volume is $2\pi^2 ab^2$.

   (b) See Figure 10. The volume of the torus is same as the volume of the torus generated by revolving the circular disc $x^2 + y^2 \leq b^2$ about the line $x = a$. Using the Shell Method, we find that the volume is $\int_{-b}^b 2\pi(a - x)(2\sqrt{b^2 - x^2})dx =$ $4\pi \left[\int_{-b}^b a\sqrt{b^2 - x^2}dx - \int_{-b}^b x(\sqrt{b^2 - x^2})dx\right] = 4\pi a \int_{-b}^b \sqrt{b^2 - x^2}dx$.

10. (a) The length of the curve is $\int_1^5 \sqrt{1 + f'(x)^2}dx = \int_1^5 (2x^2 + 1)dx$.

    (b) Since $x'(t) = -12\sin^2 t$ and $y'(t) = 12 \sin t \cos t$, the length of the curve is $\int_0^{\frac{\pi}{2}} \sqrt{(-12\sin^2 t)^2 + (12 \sin t \cos t)^2} = \int_0^{\frac{\pi}{2}} 12 \sin t dt = 12$.

    (c) The required length is $\int_0^\pi \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta = \int_0^\pi \sqrt{\sin^4 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}}d\theta$ $= \int_0^\pi \sqrt{\sin^2 \frac{\theta}{2}}d\theta = \int_0^\pi |\sin\frac{\theta}{2}|d\theta = 2$.

11. $s(x) = 2x$ implies that $\int_0^x \sqrt{1 + (f'(t)^2}dt = 2x$. By the first FTC, $f(x) = \sqrt{3}x + f(0)$.

12. See Figure 11. Note that $\int_0^\infty \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta = \int_0^\infty \sqrt{2}e^{-\theta}d\theta = \sqrt{2}$.

13. See Figure 12. Observe that $\int_0^\infty \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta = \int_0^\infty \sqrt{\frac{1}{(1+\theta)^2} + \frac{1}{(1+\theta)^4}}d\theta = \int_1^\infty \sqrt{\frac{1}{t^2} + \frac{1}{t^4}}dt$ which does not exist.