1. The curve $x = \frac{y^4}{4} + \frac{1}{8y^2}$, $1 \leq y \leq 2$, is rotated about the $y$-axis. Find the surface area of the surface generated.

2. Evaluate the area of the surface generated by revolving the curve $y = \frac{x^4}{3} + \frac{1}{4x^2}$, $1 \leq x \leq 3$, about the line $y = -2$.

3. The curve $x(t) = 2 \cos t - \cos 2t$, $y(t) = 2 \sin t - \sin 2t$, $0 \leq t \leq \pi$ is revolved about the $x$-axis. Calculate the area of the surface generated.

4. Find the area of the surface generated by revolving the curve $r = 1 + \cos \theta$, $0 \leq \theta \leq \pi$ about the $x$-axis.

5. Consider an equilateral triangle with its base lying on the $x$-axis and let $a$ be the length of its side. Using Pappus theorem, evaluate the volume of the solid generated by revolving the triangle about the line $y = -a$.

6. Using Pappus theorem evaluate the centroid of the region $D = \{(x, y) : x^2 + y^2 \leq 4, x \geq 0 \text{ and } y \geq 0\}$.

7. A regular hexagon is inscribed in the circle $(x - 2)^2 + y^2 = 1$. The hexagon is revolved about the $y$-axis. Find the surface area of the surface generated and the volume of the solid enclosed by the surface.

8. Consider the curve $C$ defined by $x(t) = \cos^3(t)$, $y(t) = \sin^3 t$, $0 \leq t \leq \frac{\pi}{2}$.

   (a) Find the length of the curve.

   (b) Find the area of the surface generated by revolving $C$ about the $x$-axis.

   (c) If $(\bar{x}, \bar{y})$ is the centroid of $C$ then find $\bar{y}$.

9. The circular disc $(x - 4)^2 + y^2 \leq 4$ is revolved about the line $y = x$. Find the volume of the solid generated.

10. Consider the semicircular arc $(x - 2)^2 + (y - 2)^2 = 4$, $y \geq 2$. The arc is rotated about the line $y + 2x = 0$. Find the area of the surface generated.

11. Let $(\bar{x}, \bar{y})$ be the centroid of the curve $y = \frac{1}{2}(x^2 + 1), 0 \leq x \leq 1$. Using Pappus theorem find $\bar{x}$.

12. **(An infinite solid (called Torricelli’s Trumpet) with finite volume enclosed by a surface with infinite surface area):**

    For $a > 1$, consider the funnel or trumpet formed by revolving the curve $y = \frac{1}{2}$, $1 \leq x \leq a$, about the $x$-axis. Let $V_a$ and $S_a$ denote respectively the volume and the surface area of the funnel. Show that $\lim_{a \to \infty} V_a = \pi$ and $\lim_{a \to \infty} S_a = \infty$.

    *(Similarly, there are curves (for example, Koch snowflake) with infinite arc lengths enclosing regions with finite areas).*
1. Surfaces area is \( f_1^2 2\pi x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dy = \int_1^2 2\pi \left( y^2 + \frac{1}{4y^2} \right) \sqrt{1 + \left( y^3 - \frac{1}{4y^2} \right)^2} dy \)
\[ = \int_1^2 2\pi \left( y^2 + \frac{1}{4y^2} \right) \left( y^3 + \frac{1}{4y^2} \right) dy. \]

2. Surfaces area is \( f_1^3 2\pi(2 + y) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx = \int_1^3 2\pi \left( \frac{x^2}{3} + \frac{1}{4x} + 2 \right) \sqrt{1 + \left( x^2 - \frac{1}{4x^2} \right)^2} dx \)
\[ = \int_1^3 2\pi \left( \frac{x^2}{3} + \frac{1}{4x} + 2 \right) (x^2 + \frac{1}{4x^2}) dx. \]

3. Observe that \( x'(t)^2 + y'(t)^2 = 8(1 - \cos t) \). The surface area is \( f_0^\pi 2\pi y(t) x'(t)^2 + y'(t)^2 dt = 2\pi f_0^\pi 2 \sin t(1 - \cos t) 2\sqrt{1 - \cos t} dt = 8\pi \sqrt{2} f_0^\pi \sin t(1 - \cos t)^{3/2} dt = \frac{12\pi a}{5}. \)

4. The surface area \( S = \int_0^b 2\pi r \theta \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} d\theta = 2\pi \int_0^\pi (1 + \cos \theta) \theta \sqrt{2(1 + \cos \theta)} d\theta = 2\pi \int_0^\pi 2 \cos^2 \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta = 32\pi \int_0^1 \cos^2 t \sin t dt \)
\[ = 32\pi \int_0^1 \cos^2 t \sin t dt. \]

5. The required volume \( V = 2\pi \rho A = 2\pi \times (a + \frac{a}{2\sqrt{3}}) \times \frac{a^2\sqrt{3}}{4}. \)

6. Since \( D \) symmetric about the line \( y = x \), the centroid lies on the line \( y = x \). Let \((\pi, \gamma)\) be the centroid. By Pappus theorem the volume generated by revolving \( D \) about the \( x \) axis is \( 2\pi \rho A \). This implies that \( 2\pi \times \gamma \times \frac{1}{4} 4\pi = \frac{16\pi a}{5} \). Therefore, the centroid is \((\frac{a}{9\pi}, \frac{8}{3\pi})\).

7. Note that, by the symmetry, the centroid of the hexagon is \((2, 0)\) (for the curve and region). By Pappus theorem, the volume \( V = 2\pi \rho A = 2\pi \times 2 \times \frac{3\sqrt{3}}{2} \) and the surface area is \( 2\pi \rho L = 2\pi \times 2 \times 6. \)

8. (a) The length \( L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^\pi \frac{\pi}{3} \cos t \sin t dt = \frac{3}{2} \int_0^\pi \frac{\pi}{3} \sin 2t dt = \frac{3}{2} \).

(b) The surface area \( S = \int_a^b 2\pi y(t) x'(t)^2 + y'(t)^2 dt = \int_0^\pi 2\pi (\sin^3 t)(3 \sin t \cos t) dt = 6\pi \int_0^\pi \sin^4 t \cos t dt \)
\[ = \frac{6\pi}{5}. \]

(c) By Pappus theorem \( S = 2\pi \gamma L \) which implies that \( \frac{6\pi}{5} = 2\pi \gamma \frac{3}{2} \). Therefore \( \gamma = \frac{3}{5}. \)

9. By Pappus theorem, the volume is \( 2\pi \rho A = 2\pi(2\sqrt{2}) (4\pi). \)

10. By Pappus theorem, the centroid of the curve is \((2, \frac{4}{2} + 2)\) and the surface area is \( 2\pi \left( \frac{6\pi + 4}{2\sqrt{2}} \right) 2\pi. \)

11. By Pappus theorem, the surface area \( S = 2\pi \rho L \) where \( S = \int_a^b 2\pi x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dy = \int_0^1 2\pi \sqrt{2y - 1} \sqrt{1 + \left( \frac{1}{2y-1} \right)^2} dy = \int_0^1 2\pi \sqrt{2y} \sqrt{\gamma dy} = \frac{2\pi}{\sqrt{2}} (2\sqrt{2} - 1) \) and \( L = \int_0^1 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx = \int_0^1 \sqrt{1 + x^2} dx = \left[ \frac{x}{2} \sqrt{1 + x^2} + \frac{1}{2} \ln(x + \sqrt{1 + x^2}) \right]_0^1 = \frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(1 + \sqrt{2}). \)

12. \( \lim_{a \to \infty} V_a = \int_0^\infty \pi \frac{1}{x^2} dx = \pi \) and \( S_a = \int_1^a 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^2}} dx \geq \int_1^a 2\pi \frac{1}{x} dx \to \infty \) as \( a \to \infty. \)