Practice Problems 22 : Areas of surfaces of revolution, Pappus Theorem

1. The curve $x = \frac{y^4}{4} + \frac{1}{8y^2}$, $1 \leq y \leq 2$, is rotated about the y-axis. Find the surface area of the surface generated.

2. Evaluate the area of the surface generated by revolving the curve $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leq x \leq 3$, about the line $y = -2$.

3. The curve $x(t) = 2\cos t - \cos 2t,\ y(t) = 2\sin t - \sin 2t,\ 0 \leq t \leq \pi$ is revolved about the x-axis. Calculate the area of the surface generated.

4. Find the area of the surface generated by revolving the curve $r = 1 + \cos\theta,\ 0 \leq \theta \leq \pi$ about the x-axis.

5. Consider an equilateral triangle with its base lying on the x-axis and let a be the length of its side. Using Pappus theorem, evaluate the volume of the solid generated by revolving the triangle about the line $y = -a$.

6. Using Pappus theorem evaluate the centroid of the region $D = \{(x, y) : x^2 + y^2 \leq 4, x \geq 0 \text{ and } y \geq 0\}$.

7. A regular hexagon is inscribed in the circle $(x - 2)^2 + y^2 = 1$. The hexagon is revolved about the y-axis. Find the surface area of the surface generated and the volume of the solid enclosed by the surface.

8. Consider the curve C defined by $x(t) = \cos^3(t),\ y(t) = \sin^3 t,\ 0 \leq t \leq \frac{\pi}{2}$.

   (a) Find the length of the curve.

   (b) Find the area of the surface generated by revolving C about the x-axis.

   (c) If $(\overline{x}, \overline{y})$ is the centroid of C then find $\overline{y}$.

9. The circular disc $(x - 4)^2 + y^2 \leq 4$ is revolved about the line $y = x$. Find the volume of the solid generated.

10. Consider the semicircular arc $(x - 2)^2 + (y - 2)^2 = 4,\ y \geq 2$. The arc is rotated about the line $y + 2x = 0$. Find the area of the surface generated.

11. Let $(\overline{x}, \overline{y})$ be the centroid of the curve $y = \frac{1}{2}(x^2 + 1), 0 \leq x \leq 1$. Using Pappus theorem find $\overline{x}$.

12. *(An infinite solid (called Torricelli's Trumpet) with finite volume enclosed by a surface with infinite surface area):*

    For $a > 1$, consider the funnel or trumpet formed by revolving the curve $y = \frac{1}{x}$, $1 \leq x \leq a$, about the x-axis. Let V_a and S_a denote respectively the volume and the surface area of the funnel. Show that $\lim_{a\to\infty} V_a = \pi$ and $\lim_{a\to\infty} S_a = \infty$.

    (Similarly, there are curves (for example, Koch snowflake) with infinite arc lengths enclosing regions with finite areas).

Practice Problems 22 : Hints/Solutions

1. Surfaces area $= \int_1^2 2\pi x\sqrt{1+(\frac{dx}{dy})^2}dy = \int_1^2 2\pi\left(\frac{y^4}{4}+\frac{1}{8y^2}\right)\sqrt{1+\left(y^3-\frac{1}{4y^3}\right)^2}dy$
$\int_1^2 2\pi\left(\frac{y^4}{4}+\frac{1}{8y^2}\right)\left(y^3+\frac{1}{4y^3}\right)dy.$

2. Surfaces area $= \int_1^3 2\pi(2+y)\sqrt{1+(\frac{dy}{dx})^2}dx = \int_1^3 2\pi\left(\frac{x^3}{3}+\frac{1}{4x}+2\right)\sqrt{1+(x^2-\frac{1}{4x^2})^2}dx$
$\int_1^3 2\pi\left(\frac{x^3}{3}+\frac{1}{4x}+2\right)(x^2+\frac{1}{4x^2})\,dx.$

3. Observe that $x'(t)^2+y'(t)^2 = 8(1-\cos t)$. The surface area is $\int_0^\pi 2\pi y(t)\sqrt{x'(t)^2+y'(t)^2}dt = 2\pi\int_0^\pi 2\sin t(1-\cos t)2\sqrt{2}\sqrt{(1-\cos t}dt = 8\pi\sqrt{2}\int_0^\pi \sin t(1-\cos t)^{\frac{3}{2}}dt = \frac{128\pi}{5}.$

4. The surface area $S = \int_a^b 2\pi r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta = 2\pi\int_0^\pi(1+\cos\theta)\sin\theta\sqrt{2(1+\cos\theta)}d\theta = 2\pi\int_0^\pi 2\cos^2\frac{\theta}{2}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}2\cos\frac{\theta}{2}d\theta = 32\pi\int_0^{\frac{\pi}{2}}\cos^4 t\sin t dt.$

5. The required volume $V = 2\pi\rho A = 2\pi\times\left(a+\frac{a}{2\sqrt{3}}\right)\times\frac{a^2\sqrt{3}}{4}.$

6. Since D symmetric about the line $y=x$, the centroid lies on the line $y=x$. Let $(\overline{x},\overline{y})$ be the centroid. By Pappus theorem the volume generated by revolving D about the x axis is $2\pi\rho A$. This implies that $2\pi\times\overline{y}\times\frac{1}{4}4\pi = \frac{16\pi}{3}$. Therefore, the centroid is $(\frac{8}{3\pi},\frac{8}{3\pi})$.

7. Note that, by the symmetry, the centroid of the hexagon is $(2,0)$ (for the curve and region). By Pappus theorem, the volume $V = 2\pi\rho A = 2\pi\times 2\times\frac{3\sqrt{3}}{2}$ and the surface area is $2\pi\rho L = 2\pi\times 2\times 6$.

8. (a) The length $L = \int_a^b\sqrt{x'(t)^2+y'(t)^2}dt = \int_0^{\frac{\pi}{2}} 3|\cos t\sin t|dt = \frac{3}{2}\int_0^{\frac{\pi}{2}}\sin 2t dt = \frac{3}{2}.$
   (b) The surface area $S = \int_a^b 2\pi y(t)\sqrt{x'(t)^2+y'(t)^2}dt = \int_0^{\frac{\pi}{2}} 2\pi(\sin^3 t)(3\sin t\cos t)dt = 6\pi\int_0^{\frac{\pi}{2}}\sin^4 t\cos t dt = \frac{6\pi}{5}.$
   (c) By Pappus theorem $S = 2\pi\overline{y}L$ which implies that $\frac{6\pi}{5} = 2\pi\overline{y}\frac{3}{2}$. Therefore $\overline{y} = \frac{2}{5}$.

9. By Pappus theorem, the volume is $2\pi\rho A = 2\pi(2\sqrt{2})(4\pi)$.

10. By Pappus theorem, the centroid of the curve is $(2,\frac{4}{\pi}+2)$ and the surface area is $2\pi(\frac{6\pi+4}{\sqrt{5}\pi})2\pi$.

11. By Pappus theorem, the surface area $S = 2\pi\overline{x}L$ where $S = \int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy = \int_{\frac{1}{2}}^1 2\pi\sqrt{2y-1}\sqrt{1+\frac{1}{2y-1}}dy = \int_{\frac{1}{2}}^1 2\pi\sqrt{2}\sqrt{y}dy = \frac{2\pi}{3}(2\sqrt{2}-1)$ and $L = \int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^1\sqrt{1+x^2}dx = \left[\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\ln(x+\sqrt{1+x^2})\right]_0^1 = \frac{1}{2}\sqrt{2}+\frac{1}{2}\ln(1+\sqrt{2}).$

12. $\lim_{a\to\infty} V_a = \int_1^\infty \pi\frac{1}{x^2}dx = \pi$ and $S_a = \int_1^a 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx \geq \int_1^a 2\pi\frac{1}{x}dx \to \infty$ as $a\to\infty$.