1. Match the parametric equation with the curve it defines. The curves are given in Figures 1-11.
   (a) \( R(t) = (t^2, t^3), t \in \mathbb{R} \) (Cuspidal cubic).
   (b) \( R(t) = (e^t \cos t, e^t \sin t), t \geq 0 \) (Logarithmic spiral)
   (c) \( R(t) = (t \cos t, t \sin t), t \geq 0 \) (Spiral)
   (d) \( R(t) = (t^2 - 1, t(t^2 - 1)), t \in \mathbb{R} \) (Crunodal cubic)
   (e) \( R(t) = (t^2 + t, 2t - 1), t \in \mathbb{R} \) (Parabola)
   (f) \( R(t) = (\cos^3 t, \sin^3 t), 0 \leq t \leq 2\pi \) (Astroid)
   (g) \( R(t) = (\sin^2 t, 2 \cos t), t \in \mathbb{R} \)
   (h) \( R(t) = (\cos t^2, \sin t^2, t^2), t \in \mathbb{R} \)
   (i) \( R(t) = (\cos t, \sin t, \sin t), t \in \mathbb{R} \)
   (j) \( R(t) = (t \cos t, t \sin t, t), t \geq 0 \)
   (k) \( R(t) = (1 + \sin t, 1 + \sin t, 2 + \sin t), t \in \mathbb{R} \)

2. Find parametric representations of the following circles.
   (a) The circle of radius 4 centered at (1,0,2) and parallel to the \( yz \)-plane.
   (b) The circle of radius 3 centered at (0,0,0) and lying on the plane containing two unit vectors \( \mathbf{u} \) and \( \mathbf{v} \) where \( \mathbf{u} \cdot \mathbf{v} = 0 \).
   (c) The circle of radius 3 centered at (1,1,2) and parallel to the plane containing two unit vectors \( \mathbf{u} \) and \( \mathbf{v} \) where \( \mathbf{u} \cdot \mathbf{v} = 0 \).
   (d) The intersection of the sphere \( x^2 + y^2 + z^2 = 4 \) and the plane \( z = y \).
   (e) The circle passing through \( e_1 = (1,0,0), e_2 = (0,1,0) \) and \( e_3 = (0,0,1) \).

3. Parameterize the curve given by \( x^3 + y^3 = 3xy \) by considering the parameter \( t = \frac{y}{x} \) which is the slope of the line through the origin and the point \((x,y)\) on the curve.

4. Consider the unit circle \( x^2 + y^2 = 1 \). By considering the parameter \( t = \frac{y}{x+y} \) which is the slope of the line joining \((1,0)\) and the point \((x,y)\) on the curve, show that \( R(t) = \left( \frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1} \right) \) is a parametric representation of the unit circle. (This parametrization of the circle is called rational parametrization).

5. Consider a parametric representation of the line \( R(t) = (x_0 + tu, y_0 + tv), t \in \mathbb{R}, (u,v) \neq (0,0) \). Show that \( vx - uy - vx_0 + uw_0 = 0 \) is an implicit equation of the line.

6. Reparameterize the following curves in terms of arc length.
   (a) \( R(t) = (2 + t, 3 - t, 5t), t \geq 0 \).
   (b) \( R(t) = (2 \cos t, 2 \sin t, \sqrt{5}t), t \geq 0 \).

7. Find two parametric representations \( R_1(t) \) and \( R_2(t) \) for the line \( y = x \) in \( \mathbb{R}^2 \) such that \( R_1(0) = R_2(0) = (0,0) \) and \( R_1'(0) \neq (0,0) \) but \( R_2'(0) = (0,0) \).

8. Consider a curve \( R(t), t \in I \) and let \( R'(t) \neq 0 \) for all \( t \). Show that the arc length parametrization \( R(t(s)) \) of the curve \( R(t) \) has unit speed, i.e., \( \frac{|dR|}{ds} = 1 \).
Practice Problems 24 : Hints/Solutions

1. The curve is sketched/identified by plotting the points \( R(t_i) \) for some \( t_1, t_2, ..., t_n \).

   (a) Note that the curve is symmetric about the \( x \) axis i.e. if \( (x(t), y(t)) \) lies on the curve then \((x(t), -y(t)) = (x(-t), y(-t))\) also lies on the curve. Moreover \( x(t) = t^2 > 0 \) for all \( t \). The curve is given in Figure 4.

   (b) Note that \( R(t) = (r(t) \cos t, r(t) \sin t), t \geq 0 \) where \( r(t) = e^t \). So \( R(t) \) is a parametric form of the polar curve \( r(t) = e^t \). The curve is given in Figure 6.

   (c) The curve is given in Figure 5. It is a polar curve \( r(t) = t, t \geq 0 \).

   (d) For \( t = 1 \) and \( t = -1 \), \( R(t) = (0, 0) \). The curve is symmetric about the \( x \)-axis. The curve is given in Figure 1.

   (e) Since \( t = \frac{y + 1}{2} \), we get \( x = \frac{y^2}{4} + y + \frac{3}{2} \) (by eliminating \( t \)). The curve is given in Figure 3.

   (f) The curve is given in Figure 2.

   (g) Note that \( 4x + y^2 = 4, 0 \leq x \leq 1 \) and \(-2 \leq y \leq 2 \). So the curve is a portion of a parabola which is given in Figure 7.

   (h) Observe that the \( x \) and \( y \) components trace out a circle in the \( xy \)-plane. The curve is given in Figure 9.

   (i) The \( x \) and \( y \) components trace out a circle and the curve lies on the plane \( z = y \). The curve is given in Figure 8.

   (j) A point \((x, y, z)\) on the curve satisfies the equation \( x^2 + y^2 = z^2 \). The curve is given in Figure 11.

   (k) If we substitute \( t' = \sin t \), the points in the curve are represented by \((1 + t', 1 + t', 2 + t')\) which lies on a straight line. Since \( \sin t \) is bounded the given curve is a line segment which is given in Figure 10.

2. (a) The given circle is a translation of the circle \( r(t) = (0, 4 \cos t, 4 \sin t) \). A parametrization of the given circle is \( R(t) = (1, 0, 2) + (0, 4 \cos t, 4 \sin t), 0 \leq t \leq 2\pi \).

   (b) Observe that any point \( p \) on the plane containing \( u, v \) and \((0, 0, 0)\) can be expressed as \( p = (p \cdot u)u + (p \cdot v)v \) (see PP 23). Let \((x, y, z)\) be a point on the circle and \( t \) be the angle between the vectors \((x, y, z)\) and \( u \). Then \((x, y, z) = (4 \cos t)u + 4(\sin t)v \). Therefore a parametric representation of the given circle is \( R(t) = (4 \cos t)u + 4(\sin t)v \).

   (c) By (b), a parametrization of the given circle is \( R(t) = (1, 1, 2) + 4(\cos t)u + 4(\sin t)v \).

   (d) Observe that the intersection is a circle lying in the plane \( z = y \) centered at \((0, 0, 0)\) with radius 2. Let \( u = (1, 0, 0) \) and \( v = \frac{1}{\sqrt{2}}(0, 1, 1) \). Then \( u \) and \( v \) are perpendicular unit vectors lying on the plane \( z = y \). Following the solution of (b), we observe that a parametric representation of the given circle is \( R(t) = (2 \cos t)(1, 0, 0) + (2 \sin t)\frac{1}{\sqrt{2}}(0, 1, 1), \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}} \).

   (e) The center of the circle is the center of the equilateral triangle formed by \( e_1, e_2 \) and \( e_3 \) which is \( u = \frac{1}{3}(1, 1, 1) \). This can be easily checked because \( \|u - e_1\| = \|u - e_2\| = \|u - e_3\| = \frac{\sqrt{2}}{3} \) and the point \( \frac{1}{3}(1, 1, 1) \) lies on the triangular region. The unit vector in the direction from the center \( u \) towards the direction of a point on the circle \( e_3 \) is \( w = \frac{1}{\sqrt{6}}(1, 1, -2) \). If \( v \) is a unit vector which is perpendicular to \( w \) and \((1, 1, 1)\) which is a normal to the plane containing \( e_1, e_2, e_3 \), then \( v = \frac{1}{\sqrt{2}}(1, -1, 0) \). Following the solution of (c), we observe that a parametric representation of the given circle is \( R(t) = \frac{1}{3}(1, 1, 1) + \frac{\sqrt{2}}{3}(\cos t)w + \frac{\sqrt{2}}{3}(\sin t)v \).
3. Substitute \( y = tx \) into the equation and get \( x = \frac{3t}{1+t^2} \), by ignoring the trivial solution \( x = 0 \). Since \( y = tx \) we get \( y = \frac{3t^2}{1+t^2} \). Therefore a parametrization for the curve is 
\[
R(t) = \left( \frac{3t}{1+t^2}, \frac{3t^2}{1+t^2} \right).
\]

4. Substitute \( y = t(x-1) \) into the equation and get \( x = \frac{t^2-1}{t^2+1} \), by ignoring the trivial solution \( x = 1 \).

5. Let \((x, y)\) be any point on the line. Then \((x-x_0, y-y_0) \times (u, v) = 0\).

6. (a) By definition \( s(t) = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2} \, d\tau = \int_0^t \sqrt{27} \, d\tau = \sqrt{27} t \). This implies that \( R(t(s)) = (2 + \frac{1}{\sqrt{27}}s, 3 - \frac{1}{\sqrt{27}}s, \frac{5}{\sqrt{27}}s) \).

   (b) By definition \( s(t) = 3t \). Therefore \( t(s) = \frac{s}{3} \) and hence \( R(t(s)) = (2 \cos \frac{s}{3}, 2 \sin \frac{s}{3}, \sqrt{5} \frac{s}{3}) \).

7. Consider \( R_1(t) = (t, t) \) and \( R_2(t) = (t^3, t^3) \), \( t \in \mathbb{R} \).

8. Since the parametrization is in terms of \( s \), \( \left\| \frac{dR}{ds} \right\| \) is the speed of \( R(t(s)) \). We know that \( \frac{dR}{ds} = T \) and therefore \( \left\| \frac{dR}{ds} \right\| = 1 \).