PP 29 : Mixed Partial Derivatives, Mean Value Theorem and Extended Mean Value theorem

The following two definitions are used in this problem sheet.

Definition 1: Let $f : \mathbb{R}^2 \to \mathbb{R}$. We say that f is convex if $f[(1-\lambda)X+\lambda Y] \leq (1-\lambda)f(X)+\lambda f(Y)$ for every $X, Y \in \mathbb{R}^2$ and every $0 \leq \lambda \leq 1$. (Geometrically, if we take two points $(X, f(X))$ and $(Y, f(Y))$ on the graph of f, then the graph of f lies below the line segment joining the two points chosen).

Definition 2: A 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be non-negative definite if the matrix multiplication $\begin{pmatrix} h & k \end{pmatrix} A \begin{pmatrix} h \\ k \end{pmatrix} = ah^2 + (b+c)hk + dk^2 \geq 0$ for all $h, k \in \mathbb{R}$.

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1. Let $f(x,y) = \frac{x^2y-y^2x}{x+y}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that, at $(0,0)$,
   (a) f is continuous.
   (b) f_x and f_y are continuous.
   (c) f is differentiable.
   (d) $f_{xy} \neq f_{yx}$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and $M \in \mathbb{R}$ be such that $|f_x(X)| \leq M$ and $|f_y(X)| \leq M$ for all $X \in \mathbb{R}^2$. Show that $|f(X) - f(Y)| \leq 2M\|X - Y\|$ for all $X, Y \in \mathbb{R}^2$.

3. *(Tangent plane approximation)*: Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$. Suppose that f_x and f_y are continuous and they have continuous partial derivatives on $\mathbb{R}^2$. Let $z = L(x,y)$ be the equation of the tangent plane for the surface $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$. Show that
   (a) $f(x,y) = L(x,y) + R$ where $R \to 0$ as $(x,y) \to (x_0, y_0)$.
   (b) $e^y \cos x = 1 + y + R$ where $R \to 0$ as $(x,y) \to (0,0)$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Show that f is convex if and only if $f(X) \geq f(X_0) + f'(X_0) \cdot (X - X_0)$ for all $X, X_0 \in \mathbb{R}^2$ (geometrically, the graph of f lies above the tangent plane at every point on the graph).

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that f_x and f_y are continuous and they have continuous partial derivatives. Then f is convex if, for all $X \in \mathbb{R}^2$, the matrix $M_X = \begin{pmatrix} f_{xx}(X) & f_{xy}(X) \\ f_{yx}(X) & f_{yy}(X) \end{pmatrix}$ is non-negative definite (See the definition given above).

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $X \in \mathbb{R}^2$. Denote $Q(X) = (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})(X)$. Show that
   (a) $f_{xx}(X)Q(X) = (hf_{xx} + kf_{xy})^2(X) + k^2(f_{xx}f_{yy} - f_{xy}^2)(X)$.
   (b) $f_{yy}(X)Q(X) = (hf_{yy} + kf_{xy})^2(X) + k^2(f_{xx}f_{yy} - f_{xy}^2)(X)$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that f_x and f_y are continuous and they have continuous partial derivatives. Show that f is convex if for all $(x,y) \in \mathbb{R}^2$ the following properties hold
   (a) $(f_{xx}f_{yy} - f_{xy})^2(x,y) \geq 0$,
   (b) $f_{xx}(x,y) \geq 0$ or $f_{yy}(x,y) \geq 0$.

8. Show that the function $f(x,y) = x^2 + y^2$ is convex.

9. (*) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second order partial derivatives. For $(x_0, y_0), (h,k) \in \mathbb{R}^2$, define

$$H(h,k) = [f(x_0+h, y_0+k) - f(x_0+h, y_0)] - [f(x_0, y_0+k) - f(x_0, y_0)]\,.$$

Show that

(a) there exists $\overline{x}$ between x_0 and x_0+h such that $H(h,k) = [f_x(\overline{x}, y_0+k) - f_x(\overline{x}, y_0)]\,h$.

(b) there exists $\overline{y}$ between y_0 and y_0+k such that $H(h,k) = f_{xy}(\overline{x}, \overline{y})hk$.

(c) $f_{xy}(x_0,y_0) = \lim_{(h,k)\to(0,0)} \frac{1}{hk} H(h,k)$.

(d) $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

Practice Problems 29: Hints/Solutions

1. (a) Note that $f(x,y) = \frac{x-y}{\frac{1}{x}+\frac{1}{y}} \to 0 = f(0,0)$ as $(x,y)\to(0,0)$.

   (b) If $(x,y) \neq (0,0)$, then $f_x(x,y) = \frac{y(x^2+2xy-y^2)}{(x+y)^2}$ and $f_y(x,y) = \frac{x(x^2-2xy-y^2)}{(x+y)^2}$. At $(0,0)$, $f_x(0,0) = f_y(0,0) = 0$. Now $|f_x(x,y)| \leq \frac{|y||x+y|^2}{|x+y|^2} = |y| \to 0$ as $(x,y)\to(0,0)$. This shows that $f_x(x,y) \to f_x(0,0)$ as $(x,y)\to(0,0)$. Therefore f_x is continuous at $(0,0)$. Similarly we show that f_y is continuous at $(0,0)$.

   (c) The differentiabilty of f at $(0,0)$ follows from (b).

   (d) By definition, $f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{-k^3}{k^3} = -1$. Similarly, verify that $f_{yx}(0,0) = 1$.

2. Follows from the mean value theorem.

3. The equation of the tangent plane is $z = L(x,y)$ where, for any $(x,y)\in\mathbb{R}^2$, $L(x,y) = f(x_0,y_0) + f'(x_0,y_0)\cdot(x-x_0, y-y_0)$.

   (a) By the EMVT there exists some C lying on the line segment joining (x,y) and (x_0,y_0) such that $f(x,y) = L(x,y) + R(x,y)$ where $R(x,y) = \frac{1}{2}[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0)f_{xy} + (y-y_0)^2 f_{yy}](C)$. By the continuity of the second order partial derivatives of f, $R(x,y)\to 0$ as $(x,y)\to(x_0,y_0)$.

   (b) Let $(x_0,y_0) = (0,0)$ and apply (a).

4. Suppose that f is convex. Let $X, X_0 \in \mathbb{R}^2$ and $\lambda\in[0,1]$. Then $f(X_0+\lambda(X-X_0)) \leq f(X_0)+\lambda(f(X)-f(X_0))$. This implies that $\frac{1}{\lambda}[f(X_0+\lambda(X-X_0))-f(X_0)] \leq f(X)-f(X_0)$. Therefore $\frac{1}{\lambda}[f(X_0+\lambda(X-X_0))-f(X_0)] - f'(X_0)\cdot(X-X_0) \leq f(X)-f(X_0)-f'(X_0)\cdot(X-X_0)$. Allow $\lambda\to 0^+$.

   Conversely, suppose that $f(X) \geq f(X_0) + f'(X_0)\cdot(X-X_0)$ for all $X, X_0\in\mathbb{R}^2$. Let $X_1, X_2 \in \mathbb{R}^2$ and $X_0 = (1-\lambda)X_1 + \lambda X_2$ for some $\lambda\in[0,1]$. Then, by the assumption, $f(X_1)-f(X_0) \geq f'(X_0)\cdot(X_1-X_0)$ and $f(X_2)-f(X_0)\geq f'(X_0)\cdot(X_2-X_0)$. From these two inequalities we get that $(1-\lambda)f(X_1)+\lambda f(X_2) - f(X_0) \geq 0$. This proves the convexity of f.

5. This follows from the EMVT and Problem 4.

6. Trivial.

7. Follows from Problem 5 and Problem 6.

8. By applying either Problem 5 or Problem 7 we see that f is convex.

9. (a) Define $g(x) = f(x, y_0 + k) - f(x, y_0)$. Then $H(h, k) = g(x_0 + h) - g(x_0)$. By the MVT (for one variable), there exists $\overline{x} \in \mathbb{R}$, between x_0 and $x_0 + h$, such that $g(x_0 + h) - g(x_0) = g'(\overline{x})h$. Note that $g'(\overline{x}) = f_x(\overline{x}, y_0 + k) - f_x(\overline{x}, y_0)$. This proves (a).

   (b) Again apply the MVT for one variable to obtain (b).

   (c) By the continuity of f_{xy}, we have $f_{xy}(x_0, y_0) = \lim_{(h,k)\to(0,0)} f_{xy}(x_0 + h, y_0 + k)$ $= \lim_{(h,k)\to(0,0)} f_{xy}(\overline{x}, \overline{y})$. Apply (b).

   (d) By exchanging the rolls of x and y, we show that $f_{yx}(x_0, y_0) = \lim_{(h,k)\to(0,0)} \frac{1}{hk} H(h, k)$.