Practice Problems 3 : Cauchy criterion, Subsequence

1. Show that the sequence (x_n) defined below satisfies the Cauchy criterion.

   (a) $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$ for all $n \geq 1$

   (b) $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n^2}$ for all $n \geq 1$.

   (c) $x_1 = 1$ and $x_{n+1} = \frac{1}{6}(x_n^2 + 8)$ for all $n \geq 1$.

2. Let (x_n) be a sequence of positive real numbers. Prove or disprove the following statements.

   (a) If $x_{n+1} - x_n \to 0$ then (x_n) converges.

   (b) If $|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$ then (x_n) converges.

   (c) If (x_n) satisfies the Cauchy criterion, then there exists an $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$ and $|x_{n+1} - x_n| \leq \alpha |x_n - x_{n-1}|$ for all $n \in \mathbb{N}$.

3. Let (x_n) be a sequence of integers such that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Prove or disprove the following statements.

   (a) The sequence (x_n) does not satisfy the Cauchy criterion.

   (b) The sequence (x_n) cannot have a convergent subsequence.

4. Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence satisfying the condition: $|x_{n+1} - x_n| \leq \alpha^n, \quad n = 1, 2, 3, \ldots$. Show that (x_n) satisfies the Cauchy criterion.

5. Let (x_n) be defined by: $x_1 = \frac{1}{1!}, x_2 = \frac{1}{1!} - \frac{1}{2!}, \ldots, x_n = \frac{1}{1!} - \frac{1}{2!} + \ldots + \frac{(-1)^{n+1}}{n!}$. Show that the sequence converges.

6. Let $1 \leq x_1 \leq x_2 \leq 2$ and $x_{n+2} = \sqrt{x_{n+1}x_n}$, $n \in \mathbb{N}$. Show that $\frac{x_{n+1}}{x_n} \geq \frac{1}{2}$ for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq \frac{2}{3}|x_n - x_{n-1}|$ for all $n \in \mathbb{N}$ and (x_n) converges.

7. **(*)** Show that a sequence (x_n) of real numbers has no convergent subsequence if and only if $|x_n| \to \infty$.

8. **(*)** Let (x_n) be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of (x_n) has a convergent subsequence converging to x_0. Show that $x_n \to x_0$.

9. **(*)** Let (x_n) be a sequence in $\mathbb{R}$. We say that a positive integer n is a peak of the sequence if $m > n$ implies $x_n > x_m$ (i.e., if x_n is greater than every subsequent term in the sequence).

   (a) If (x_n) has infinitely many peaks, show that it has a decreasing subsequence.

   (b) If (x_n) has only finitely many peaks, show that it has an increasing subsequence.

   (c) From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, every bounded sequence in $\mathbb{R}$ has a convergent subsequence (An alternate proof of Bolzano-Weierstrass Theorem).

Hints/Solutions

1. (a) Note that $|x_{n+1}-x_n| = |\frac{1}{x_n} - \frac{1}{x_{n-1}}| = |\frac{x_{n-1}-x_n}{x_n x_{n-1}}|$ and $|x_n x_{n-1}| = |(1+\frac{1}{x_{n-1}})x_{n-1}| = |x_{n-1}+1| \geq 2$. This implies that $|x_{n+1}-x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$. Hence (x_n) satisfies the contractive condition and therefore it satisfies the Cauchy criterion.

   (b) Observe that $|x_{n+1}-x_n| = \frac{|x_n^2 - x_{n-1}^2|}{(2+x_n^2)(2+x_{n-1}^2)} \leq \frac{|x_n - x_{n-1}||x_n + x_{n-1}|}{4} \leq \frac{2}{4}|x_n - x_{n-1}|$.

   (c) We have $|x_{n+1}-x_n| \leq \frac{|x_n - x_{n-1}||x_n + x_{n-1}|}{6} \leq \frac{4}{6}|x_n - x_{n-1}|$.

2. (a) False. Choose $x_n = \sqrt{n}$ and observe that $x_{n+1} - x_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$.

   (b) False. For $x_n = \sqrt{n}$, $|x_{n+2}-x_{n+1}| = |\sqrt{n+2}-\sqrt{n+1}| < \frac{1}{\sqrt{n+1}+\sqrt{n}} = |x_{n+1}-x_n|$.

   (c) False. Take $x_n = \frac{1}{n}$. If $|\frac{1}{n+1} - \frac{1}{n}| \leq \alpha|\frac{1}{n} - \frac{1}{n-1}|$ for some $\alpha > 0$, show that $\alpha \geq 1$.

3. (a) True. Because $|x_{n+1} - x_n| \nrightarrow 0$ as $n \to \infty$.

   (b) False. Consider $x_n = (-1)^n$.

4. Let $n > m$. Then $|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$
   $\leq \alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m = \alpha^m[1 + \alpha + \cdots + \alpha^{n-1+m}] \leq \frac{\alpha^m}{1-\alpha} \to 0$ as $m \to \infty$.

   Thus (x_n) satisfies the Cauchy criterion.

5. Use Problem 4.

6. Since $1 \leq x_n \leq 2$, $\frac{x_{n+1}}{x_n} \geq \frac{1}{2}$. Observe that $x_{n+1}^2 - x_n^2 = x_n x_{n-1} - x_n^2 = x_n(x_{n-1} - x_n)$. Therefore $|x_{n+1} - x_n| = |\frac{x_n}{x_{n+1}+x_n}||x_{n-1} - x_n| \leq \frac{2}{3}|x_n - x_{n-1}|$.

7. Suppose $|x_n| \to \infty$. If (x_{n_k}) is a subsequence of (x_n), then observe that $|x_{n_k}| \to \infty$. If $|x_n| \nrightarrow \infty$, then there exists a bounded subsequence of (x_n). Apply Bolzano-Weierstrass theorem.

8. Suppose $x_n \nrightarrow x_0$. Then there exists $\epsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x_0| \geq \epsilon_0$ for all n_k. Note that (x_{n_k}) has no subsequence converging to x_0.

9. (a) If (x_n) has infinitely many peaks, $n_1 < n_2 < ... < n_j < ....$ Then the subsequence (x_{n_j}) is decreasing.

   (b) Suppose there are only finite peaks and let N be the last peak. Since $n_1 = N+1$ is not a peak, there exists $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$. Again $n_2 > N$ is not a peak, there exists $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$. Continuing this process we find an increasing sequence (x_{n_k}).