PP 31 : Method of Lagrange Multipliers

1. Using the method of Lagrange multipliers, find three real numbers such that the sum of the numbers is 12 and the sum of their squares is as small as possible.

2. Let $f(x,y) = \frac{x^2}{16} + \frac{y^2}{4\pi}$, $g(x,y) = x + y - 1$ for all $(x,y) \in \mathbb{R}^2$ and $D = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$.

   (a) Show that the minimum value of f on D exists.

   (b) Find the minimum value of f on D by reducing the problem to an unconstrained problem in one variable.

   (c) Does the maximum value of f on D exist?

   (d) Find the minimum value of f on D by the method of Lagrange multipliers.

   (e) A wire of length 1 meter is cut into two parts. A square and circle are formed with the two pieces by bending them. Find the least value of the sum of the areas of the square and the circle formed. Discuss whether the maximum value of the sum exists.

3. Let $f(x,y) = x^2 + y^3$ and $g(x,y) = x^4 + y^6 - 2$.

   (a) Find the set of points satisfying the Lagrange system of equations $\bigtriangledown f = \lambda \bigtriangledown g$ and $g = 0$.

   (b) Show that f achieves its maximum and minimum on the set $\{(x,y) : g(x,y) = 0\}$.

   (c) Find the points of maxima and minima of the function f subject to the constraint $g(x,y) = 0$.

4. Let $f(x,y,z) = xyz$ and $g(x,y,z) = x^2 + y^2 + z^2 - 12$.

   (a) Find the set of points (x,y,z) satisfying the equations $\nabla f = \lambda \nabla g$ and $g = 0$ when $\lambda = 0$ and $\lambda \neq 0$.

   (b) Find the maximum and minimum values of f subject to the constraint $g = 0$.

5. Find the maximum and minimum values of the function $f(x,y,z) = x^2+y^2+z^2-4(x+y+z)$ on $D = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 16, z \geq 0\}$.

6. Let a, b, c and d be positive real numbers. Find the distance from the origin to the plane $ax + by + cz = d$ using

   (a) the distance formula

   (b) the method of Lagrange multipliers.

7. Suppose that a, b and c are the lengths of the sides of a triangle and A be its area. Let P be a point inside the triangle such that the sum of the distances from P to the sides of the triangles minimum. Express this minimum in terms of a, b, c and A.

8. (a) For $n \geq 2$, let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x_1, x_2, ..., x_2) = x_1^2 x_2^2 ... x_n^2$ and $c \in \mathbb{R}$. Find the maximum value of f subject to the constraint $g(x_1, x_2, ..., x_n) = x_1^2 + x_2^2 + ... + x_n^2 - c^2 = 0$.

   (b) Using (a), prove the inequality $(a_1 a_2 ... a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + ... + a_n}{n}$ for any positive real numbers $a_1, a_2, ..., a_n$.

Practice Problems 31: Hints/Solutions

1. We minimize the function $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $g(x,y,z) = x+y+z-12 = 0$. Solving the system of equations $\bigtriangledown f = \lambda \bigtriangledown g$ and $g = 0$ imply that $2x = 2y = 2z = \lambda$ and $\frac{3\lambda}{2} = 12$. Therefore we get $\lambda = 8$ and $x = y = z = 4$. Note that the function f achieves its minimum on the set $\{(x,y,z) : g(x,y,z) = 0\}$ because the point of minimum is the point on the plane $x+y+z = 12$ which is nearest to the origin. Therefore the required three real numbers are x, y and z where $x = y = z = 4$.

2. (a) Let L denote the line segment joining $(0,1)$ and $(1,0)$. Then $f(x,y) \geq f(1,0) = \frac{1}{16}$ for all $(x,y) \in D \setminus L$. Since f is continuous on the closed and bounded subset L, it achieves its minimum on L and hence on D.

   (b) Since $g(x,y) = 0$ implies $y = 1-x$, we minimize the unconstrained function $f_1(x) = f(x, 1-x) = \frac{1}{16\pi}[(\pi+4)x^2 - 8x + 4], x \in \mathbb{R}$. By the first and second derivative tests for one variable we see that $x = \frac{4}{4+\pi}$ is a point of minimum for f_1 and hence $f(\frac{4}{4+\pi}, 1 - \frac{4}{4+\pi})$ is the minimum value of f on D.

   (c) If $(x,y) \in D$ and $\|(x,y)\| \to \infty$ then $f(x,y) \to \infty$. Therefore maximum value of f on D does not exists.

   (d) Solving the system of equations $\bigtriangledown f = \lambda \bigtriangledown g$ and $g = 0$ imply that $\lambda = \frac{1}{8+2\pi}$ and $(8\lambda, 2\pi\lambda)$ satisfies the equations. Since f achieves its minimum value on D, the point $f(\frac{4}{4+\pi}, \frac{\pi}{4+\pi})$ is the point of minimum and $f(\frac{4}{4+\pi}, \frac{\pi}{4+\pi})$ is the minimum value of the function f on D.

   (e) If x and y are the lengths (in meters) of the pieces which are used to form a square and a circle respectively. The problem is reduced to minimizing the function $f(x,y) = \frac{x^2}{16} + \frac{y^2}{4\pi}$ on the set $T = \{(x,y,z) : x+y-1 = 0, x > 0 \text{ and } y > 0\}$. From (b) or (d) it is clear that $f(\frac{4}{4+\pi}, \frac{\pi}{4+\pi})$ is the minimum value of the function f on T. From the solution of (b) we observe that the largest value of f on T is either $f(0,1)$ or $f(1,0)$. Since the points $(0,1)$ or $(1,0)$ do not lie on the set T, the maximum value of f on T does not exist.

3. The Lagrange system of equations are $x^4 + y^6 = 2$, $2x = 4\lambda x^3$ and $3y^2 = 6\lambda y^5$.

   (a) By taking $x = 0$ in the second and first equations we obtain the points $(0, \pm\sqrt[6]{2})$ which satisfy the system of equations (for some λ). By taking $y = 0$ in the third and first equations, we get the points $(\pm\sqrt[4]{2}, 0)$. If $x, y \neq 0$, from the second and the third equations we get $\lambda = \frac{1}{2x^2} = \frac{1}{2y^3}$ which imply that $x^2 = y^3$. Substituting $x^2 = y^3$ in the first equation, we obtain either $x^4 = 1$ or $y^6 = 1$ which implies either $x \pm 1$ or $y = \pm 1$. Since $x^2 = y^3$, the points $(\pm 1, 1)$ satisfy the system of equations.
   Therefore $(0, \pm\sqrt[6]{2})$,$(\pm\sqrt[4]{2}, 0)$ and $(\pm 1, 1)$ satisfy the system of equations.

   (b) Note that the set $D = \{(x,y) : g(x,y) = 0\} \subset \{(x,y) : x \in [-2,2] \text{ and } y \in [-2,2]\}$. Since f is continuous on the closed and bounded set D, it achieves its maximum and minimum on D.

   (c) Comparing the values of f at the six points mentioned above we obtain that $(0, -\sqrt[6]{2})$ is a point of minimum and $(\pm 1, 1)$ are the points of maxima.

4. (a) In the case of $\lambda = 0$, the six points $(\pm\sqrt{12}, 0, 0)$,$(0, \pm\sqrt{12})$ and $(0, 0, \pm\sqrt{12})$ satisfy the equations.
   When $\lambda \neq 0$, the equations imply that $3xyz = 2\lambda(x^2+y^2+z^2)$ which imply that $xyz = 8\lambda$. Since $x, y, z \neq 0$ and $\lambda \neq 0$, we get $2z\lambda z = 2y\lambda y = 2x\lambda x = 8\lambda$,. Therefore the eight points $(\pm 2, \pm 2, \pm 2)$ satisfy the equations.

(b) Note that the continuous function f achieves its maximum and minimum on the sphere $x^2 + y^2 + z^2 = 12$. Comparing the values of f at the fourteen points mentioned above we see that -8 and 8 are the required minimum and maximum values respectively.

5. The maximum and minimum values of f on D exist as f is continuous on a closed bounded set D. Note that $D = D_1 \cup D_2 \cup D_3 \cup D_4$ where $D_1 = \{(x,y,z) : x^2+y^2+z^2 < 16, z > 0\}$, $D_2 = \{(x,y,z) : x^2 + y^2 + z^2 = 16, z > 0\}$, $D_3 = \{(x,y,0) : x^2 + y^2 < 16\}$ and $D_4 = \{(x,y,0) : x^2+y^2 = 16\}$. We consider f on $D_i, i = 1,2,3,4$. On D_1 and D_3 we use the first derivative test to find the critical points and D_2 and D_4 we use the method of Lagrange multiplier.

   On D_1 : Solving $f_x = f_y = f_z = 0$ implies the critical point $(2,2,2)$.

   On D_3 : The function is $f(x,y,0) = x^2 + y^2 - 4(x+y)$. Solving $f_x = f_y = 0$ implies the critical point $(2,2,0)$.

   On D_2 : Solving the system of equations: $\nabla f = \lambda \nabla g$ and $g = 0$ where $g(x,y,z) = x^2 + y^2 + z^2 - 16$, implies $\lambda = \frac{2\pm\sqrt{3}}{2}$ and a corresponding $(x,y,z) = (\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}})$.

   On D_4 : Solving the system of equations: $\nabla f_1 = \lambda \nabla g_1$ and $g_1 = 0$, where $f_1(x,y,z) = x^2 + y^2 - 4(x+y)$ and $g_1(x,y,z) = x^2 + y^2 - 16$, implies $\lambda = \frac{2\pm\sqrt{2}}{2}$ and the corresponding points $(-2\sqrt{2}, -2\sqrt{2}, 0)$ and $(2\sqrt{2}, 2\sqrt{2}, 0)$.

   Comparing the values of f at $(2,2,2)$, $(2,2,0)$, $(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}})$, $(-2\sqrt{2}, -2\sqrt{2}, 0)$ and $(2\sqrt{2}, 2\sqrt{2}, 0)$ we see that the maximum value of f is $f(-2\sqrt{2}, -2\sqrt{2}, 0) = 16(\sqrt{2}+1)$ and the minimum value $f(2,2,2) = -12$.

6. (a) By the distance formula the required distance is $\frac{d}{\sqrt{a^2+b^2+c^2}}$.

   (b) If (x,y,z) is in the plane then the distance between the origin and (x,y,z) is $\sqrt{x^2+y^2+z^2}$. To find the minimum distance we minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = ax+by+cz-d = 0$. The equations $\nabla f = \lambda \nabla g$ and $g = 0$ imply that $\lambda = \frac{2d}{a^2+b^2+c^2}$, $x_0 = \frac{da}{a^2+b^2+c^2}$, $y_0 = \frac{db}{a^2+b^2+c^2}$ and $z_0 = \frac{dc}{a^2+b^2+c^2}$ satisfy the above equations. Since the minimum distance is achieved, the point (x_0, y_0, z_0) is the point of minimum for f subject to $g = 0$. Therefore the minimum distance is $\frac{d}{\sqrt{a^2+b^2+c^2}}$.

7. Let R, S and T be the vertices of the given triangle and a, b and c be the lengths of ST, RT and RS. Let x, y and z be the distance between P to ST, RT and RS respectively. Now the problem is reduced to minimizing the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = ax + by + cz - 2A = 0$. The rest follows from Problem 6.

8. (a) The function f is continuous and the set $\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : g(x_1, x_2, ..., x_n) = 0\}$ is bounded in $\mathbb{R}^n$. Therefore f achieves its maximum and minimum on the given constraint set. The equations $\nabla f = \lambda \nabla g$ and $g = 0$ imply that $2x_1x_2^2...x_n^2 = \lambda 2x_1, 2x_1^2x_2...x_n^2 = \lambda 2x_2, ..., 2x_1^2x_2^2...x_n = \lambda 2x_n$ and $g = 0$. For $\lambda \neq 0$, we have $2\lambda x_1^2 = 2\lambda x_2^2 = ... = 2\lambda x_n^2$ and $g = 0$ which imply that $x_i = \pm\frac{c}{\sqrt{n}}$ for $i = 1, 2, ..., n$. Therefore the required maximum value is $\frac{c^{2n}}{n^n}$. Observe that the minimum value of the function is 0 which is captured by $\lambda = 0$.

   (b) In (a), take $x_1^2 = a_1, x_2^2 = a_2, ..., x_n^2 = a_n$ and $x_1^2 + x_2^2 + ... + x_n^2 = c^2$ for some c. We obtain that $(a_1a_2...a_n)^{\frac{1}{n}} \leq \frac{c^2}{n} = \frac{x_1^2+x_2^2+...+x_n^2}{n} = \frac{a_1+a_2+...+a_n}{n}$.