The plane curve $C$ described in this problem sheet is oriented counterclockwise.

1. Evaluate the line integral

$$\oint_C (x^2 \sin^2 x - y^3) \, dx + (y^2 \cos^2 y - y) \, dy$$

where $C$ is the closed curve consisting of $x + y = 0$, $x^2 + y^2 = 25$ and $y = x$ and lying in the first and fourth quadrant.

2. Let a square $R$ be enclosed by $C$ and

$$\oint_C (xy^2 + x^3 \sin x) \, dx + (x^2 y + 2x) \, dy = 6.$$ 

Find the area of the square.

3. Let $C$ be a simple closed smooth curve and $\alpha$ be a real number. Suppose

$$\oint_C (\alpha e^x y + e^x) \, dx + (e^x + ye^y) \, dy = 0.$$ 

Find $\alpha$.

4. Let $D$ be the region enclosed by a simple closed piecewise smooth curve $C$. Let $F$, $F_x$ and $F_y$ be continuous on an open set containing $D$. Show that

$$\iint_D F_x \, dx \, dy = \oint_C F \, dy \quad \text{and} \quad \iint_D F_y \, dx \, dy = -\oint_C F \, dx.$$ 

5. Let $C$ be the ellipse $x^2 + xy + y^2 = 1$. Evaluate $\oint_C (\sin y + x^2) \, dx + (x \cos y + y^2) \, dy$.

6. Let $D$ be the region enclosed by a simple closed smooth curve $C$. Show that

$$\text{Area of } D = \oint_C x \, dy = -\oint_C y \, dx.$$ 

7. Evaluate the area of the region enclosed by the simple closed curve $x^{2/3} + y^{2/3} = 1$.

8. Find the area between the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the circle $x^2 + y^2 = 25$.

9. Let $f : [a, b] \to \mathbb{R}$ be a non-negative function such that its first derivative is continuous. Suppose $C$ is the boundary of the region bounded above by the graph of $f$, below by the $x$-axis and on the sides by the lines $x = a$ and $x = b$. Show that

$$\int_a^b f(x) \, dx = -\oint_C y \, dx.$$ 

10. Let $D$ be the region enclosed by the rays $\theta = a$, $\theta = b$ and the curve $r = f(\theta)$. Use Green’s theorem to derive the formula

$$A = \frac{1}{2} \int_a^b r^2 \, d\theta$$

for the area of $D$. 
1. Let \( R \) be the region enclosed by \( C \) (see Figure 1). By Green’s theorem
\[
\oint_C (x^2 \sin^2 x - y^3)dx + (y^2 \cos^2 y - y)dy = \int \int_R 3y^2 dxdy = \int_{-\pi/4}^{\pi/4} \int_0^5 3r^3 \sin^2 \theta drd\theta.
\]

2. By Green’s theorem,
\[
\oint_C (xy^2 + x^3 \sin^3 x)dx + (x^2 y + 2x)dy = \int \int_R 2 = 6.
\]
The area of \( R \) is 3.

3. Let \( R \) be the region enclosed by \( C \). By Green’s theorem,
\[
\oint_C (\alpha e^x y + e^x)dx + (e^x + ye^y)dy = \int \int_R (1 - \alpha)e^x dxdy = 0.
\]
Hence \( \alpha = 1 \).

4. Follows from Green’s theorem.

5. By Green’s theorem, \( \oint_C (\sin y + x^2)dx + (x \cos y + y^2) = 0 \).

6. Follows from Green’s theorem.

7. Let \( C \) denote the curve (see Figure 2). Then \( C \) is parameterized as \( x(\theta) = \cos^3 \theta \) and \( y(\theta) = \sin^3 \theta \), \( 0 \leq \theta \leq 2\pi \). The required area is
\[
A = \frac{1}{2} \oint_C xdy - ydx = \frac{3}{2} \int_0^{2\pi} \sin^2 \theta \cos^2 \theta d\theta = \frac{3}{8} \int_0^{2\pi} \sin^2 2\theta d\theta = \frac{3}{8} \pi.
\]

8. The circle and the ellipse are parameterized as
\[
x_1(\theta) = (5 \cos \theta, 5 \sin \theta) \quad \text{and} \quad x_2(\theta) = (2 \sin \theta, 3 \cos \theta), \quad \theta \in [0, 2\pi],
\]
(see Figure 3). The require area is \( A = \frac{1}{2} \int_0^{2\pi} [(5 \cos \theta)(5 \cos \theta) + (5 \sin \theta)(5 \sin \theta)]d\theta \)
\[
+ \frac{1}{2} \int_0^{2\pi} [-(2 \sin \theta)(3 \sin \theta) + (-3 \cos \theta)(2 \cos \theta)]d\theta = 19\pi.
\]

9. Follows from Problem 6 (See Figure 4).

10. Parameterize the rays and the curve as follows (see Figure 5):
\[
C_1 := (r \cos a, r \sin a), \quad 0 \leq r \leq f(a),
\]
\[
C_2 := (r(\theta) \cos \theta, r(\theta) \sin \theta), \quad a \leq \theta \leq b,
\]
\[
C_3 := (r \cos b, r \sin b), \quad 0 \leq r \leq f(b).
\]
The required area is \( \frac{1}{2}(\oint_{C_1} + \oint_{C_2} - \oint_{C_3})\{xdy - ydx\} \).