1. Let $D$ be the solid bounded by $z = 0$ and the paraboloid $z = 4 - x^2 - y^2$. Let $S$ be the boundary of $D$. If

$$F(x, y, z) = (x^3 + \cos(yz), y^3, x + \sin(xy)),$$

find $\iint_{S} F \cdot \hat{n} d\sigma$ where $\hat{n}$ is the unit outward normal to the surface $S$.

2. Let $S$ be the sphere $x^2 + y^2 + z^2 = 1$. Evaluate the surface integral

$$\iint_{S} [x(2x + 3e^{z^2}) + y(-y - e^{z^2}) + z(2z + \cos^2 y)]d\sigma.$$ 

3. Let $S$ be the sphere $x^2 + y^2 + (z - 1)^2 = 9$. Find the unit outward normal to the surface $S$ and evaluate the surface integral

$$\iint_{S} [x^2 \sin y + y \cos^2 x + (z - 1)(y^2 - z \sin y)]d\sigma.$$ 

4. Let $D$ be the region enclosed by the surfaces $x^2 + y^2 = 4$, $z = 0$ and $z = x^2 + y^2$. Let $S$ be the boundary of $D$ and $\hat{n}$ denote the unit outward normal vector to $S$. Suppose $F$ is a vector field whose components have continuous first order partial derivatives. If $\text{div} F = \alpha (x - 1)$ for some $\alpha \in \mathbb{R}$ and $\iint_{S} F \cdot \hat{n} d\sigma = \pi$, evaluate $\alpha$.

5. Let $S$ be the sphere $x^2 + y^2 + z^2 = 1$. Suppose for some $\alpha \in \mathbb{R}$, $\iint_{S} [xz + \alpha y^2 + xz]d\sigma = \frac{4\pi}{3}$. Find $\alpha$.

6. Let $S$ be the hemisphere $x^2 + y^2 + z^2 = 1$ and $z \geq 0$. Evaluate $\iint_{S} [(z + \cos z)x + y^2 + xz]d\sigma$.

Practice Problems 39: Hints/Solutions

1. By divergence theorem

$$\iint_{S} F \cdot \hat{n} d\sigma = \iiint_{D} \text{div} F dV = \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{3r^2} 3r^2 zdrd\theta = 32\pi.$$ 

2. Observe that the given surface integral is $\iint_{S} F \cdot \hat{n} d\sigma$ where $F(x, y, z) = (2x + 3e^2, -y - e^2, 2z + \cos^2 y)$ and $\hat{n} = (x, y, z)$ which is the unit outward normal to the sphere. By divergence theorem $\iint_{S} F \cdot \hat{n} d\sigma = \iiint_{D} \text{div} F dV = 3 \iiint_{D} dV = 4\pi$.

3. The given surface is $g(x, y, z) = 9$ where $g(x, y, z) = x^2 + y^2 + (z - 1)^2$. The unit normal vector $\hat{n}$ of $S$ is $\frac{\nabla g}{\|\nabla g\|} = \frac{1}{3}(x, y, z - 1)$. Verify that $\hat{n}$ is the unit outward normal vector. The given surface integral is $\iint_{S} F \cdot \hat{n} d\sigma$ where $F(x, y, z) = (x \sin y, \cos x, y^2 - z \sin y)$. By divergence theorem, $\iint_{S} F \cdot 3\hat{n} d\sigma = 3 \iiint_{D} \text{div} F dV = 0$.

4. By divergence theorem $\iint_{S} F \cdot \hat{n} d\sigma = \iiint_{D} \alpha (x - 1) dV = \alpha \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{r^2} (r \cos \theta - 1)dzdrd\theta = -8\pi \alpha$. Therefore $\alpha = -\frac{1}{8}$.

5. Let $D$ denote the solid enclosed by the surface $S$. By divergence theorem, $\iint_{S}(z, \alpha y, x) \cdot (x, y, z)d\sigma = \iiint_{D} \alpha dV = \alpha \frac{4\pi}{3}$. Hence $\alpha = 1$.

6. Let $F(x, y, z) = (z + \cos z, y, x)$ and $S_1$ be the disk $x^2 + y^2 \leq 1$, $z = 0$. Note that $S$ is not a closed surface. Suppose $D$ denotes the solid $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$. By divergence theorem, $\iint_{S}[(z + \cos z)x + y^2 + xz]d\sigma = \iiint_{D} \text{div} F dV - \iint_{S_1}(z + \cos z, y, x) \cdot (-\hat{k}) d\sigma = \frac{2\pi}{3}$. 

PP 39 : Divergence Theorem