Practice problems 4 : Continuity and Limit

1. Find the value of $\alpha$ such that \( \lim_{x \to -1} \frac{2x^2 - \alpha x - 14}{x^2 - 2x - 3} \) exists. Find the limit.

2. Let \( \lim_{x \to 0} \frac{f(x)}{x} = 5 \). Show that \( \lim_{x \to 0} \frac{f(x)}{x} = 0 \).

3. Let \( f : \mathbb{R} \to \mathbb{R} \) and \( x_0 \in \mathbb{R} \). Suppose \( \lim_{x \to x_0} f(x) \) exists. Show that \( \lim_{x \to x_0} f(x + x_0) = \lim_{x \to x_0} f(x) \).

4. Let \( f(x) = |x| \) for every \( x \in \mathbb{R} \). Show that \( f \) is continuous on \( \mathbb{R} \).

5. Let \( f : [0, \pi] \to \mathbb{R} \) be defined by \( f(0) = 0 \) and \( f(x) = x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \) for \( x \neq 0 \). Is \( f \) continuous?

6. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be continuous such that given any two points \( x_1 < x_2 \), there exists a point \( x_3 \) such that \( x_1 < x_3 < x_2 \) and \( f(x_3) = g(x_3) \). Show that \( f(x) = g(x) \) for all \( x \).

7. Let \( f(x) = 0 \) when \( x \) is rational and 1 when \( x \) is irrational. Determine the points of continuity for the function \( f \).

8. Let \([\cdot]\) denote the integer part function and \( f : [0, \infty) \to \mathbb{R} \) be defined by \( f(x) = [x^2] \sin \pi x \). Show that \( f \) is continuous at each \( x \neq \sqrt{n}, \ n = 1, 2, \ldots \). Further, show that \( f \) is discontinuous on \( \{x \in [0, \infty) : x = \sqrt{n} \ where \ n \neq k^2, \ for \ some \ positive \ integer \ k \} \).

9. Let \( f : \mathbb{R} \to (0, \infty) \) satisfy \( f(x + y) = f(x)f(y) \) for all \( x, y \in \mathbb{R} \). Suppose \( f \) is continuous at \( x = 0 \). Show that \( f \) is continuous at all \( x \in \mathbb{R} \).

10. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( f(x) = f(x^2) \) for all \( x \in \mathbb{R} \). Show that \( f \) is constant.

11. Suppose \( f : [0, \infty) \to \mathbb{R} \) is continuous and \( \lim_{x \to \infty} f(x) \) exists. Show that \( f \) is bounded on \([0, \infty)\).

12. (*) Let \( f : [0, 1] \to \mathbb{R} \) be one-one. Suppose \( f \) is continuous. Show that \( f^{-1} \) is also continuous on \( \{f(x) : x \in [0, 1]\} \), the range of \( f \).

13. (*) Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and \( f(x + y) = f(x) + f(y) \) for all \( x, y \in \mathbb{R} \). Show that \( f(x) = f(1)x \) for all \( x \in \mathbb{R} \).

14. (*) Let \( f : (0, 1) \to \mathbb{R} \) be given by

   \[
   f(x) = \begin{cases} 
   \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factors} \\
   0 & \text{if } x \text{ is irrational}
   \end{cases}
   \]

   (a) Let \( x_n = \frac{p_n}{q_n} \in (0, 1) \) where \( p_n, q_n \in \mathbb{N} \) and have no common factors. Suppose \( x_n \to x \) for some \( x \) with \( x_n \neq x \) for all \( n \in \mathbb{N} \). Show that \( \lim_{n \to \infty} q_n = \infty \).

   (b) Show that \( f \) is continuous at every irrational.

   (c) Show that \( f \) is discontinuous at every rational.


1. \( \alpha = 12 \) and the limit is 4.

2. Note that \( \frac{f(x)}{x} = \frac{f(x)}{x^2}x \) for \( x \neq 0 \).

3. Let \( \lim_{x \to x_0} f(x) = M \) for some \( M \in \mathbb{R} \). Let \( x_n \to 0 \), \( x_n \neq 0 \) \( \forall n \). Then \( x_n + x_0 \to x_0 \).
   Since \( \lim_{x \to x_0} f(x) = M, f(x_n + x_0) \to M \). This implies that \( \lim_{x \to 0} f(x + x_0) = M \).

4. Let \( x \in \mathbb{R} \) and \( x_n \to x \). Then \( |x_n| \to |x| \), because, \( |x_n - |x|| \leq |x_n - x| \). Therefore \( f \) is continuous at \( x \).

5. The function is not continuous at 0, because, \( x_n = \frac{1}{2n} \to 0 \) but \( f(\frac{1}{2n}) \to f(0) \).

6. Fix some \( x_0 \in \mathbb{R} \). For every \( n \), find \( x_n \) such that \( x_0 - \frac{1}{n} < x_n < x_0 \) and \( (f - g)(x_n) = 0 \).
   Allow \( n \to \infty \) and apply the continuity.

7. Suppose \( x_0 \) is rational. Find an irrational sequence \( (x_n) \) such that \( x_n \to x_0 \). Then \( f(x_n) = 1 \to f(x_0) = 0 \). Therefore \( f \) is not continuous at \( x_0 \). Let \( y_0 \) be rational. Show that \( f \) is not continuous at \( y_0 \).

8. Case 1: \( x_0 \neq \sqrt{n}, n = 1, 2, \ldots \) It is clear that \( f \) is continuous at \( x_0 \). Case 2: \( x_0 = \sqrt{n} \) where \( n = k^2 \), for some positive integer \( k \). In this case \( \lim_{x \to k^2} f(x) = \lim_{x \to k} f(x) = 0 \). Case 3: \( x_0 = \sqrt{n} \) where \( n \neq k^2 \), for some positive integer \( k \). In this case, \( \lim_{x \to \sqrt{n}} f(x) = n \sin(\pi \sqrt{n}) \) and \( \lim_{x \to -\sqrt{n}} f(x) = (n - 1) \sin(\pi \sqrt{n}) \).

9. Since \( f(0) = f(0)^2, f(0) = 1 \) and since \( f(x - x) = f(0), f(-x) = \frac{1}{f(x)} \). Let \( x_0 \in \mathbb{R} \) and \( x_n \to x_0 \). By continuity at 0, \( f(x_n - x_0) \to 1 \) and hence \( f(x_n) \to \frac{1}{f(0)} = f(x_0) \).

10. Suppose \( x > 0 \). By the assumption, \( f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{2^n}}) = f(x^{\frac{1}{2^n}}) \). Since \( x^{\frac{1}{2^n}} \to 1, f(x^{\frac{1}{2^n}}) \to f(1), i.e. f(x) = f(1). \) Now \( f(-x) = f((-x)^2) = f(x^2) = f(x) \). At \( x = 0 \), by continuity, \( \lim_{x \to 0} f(x) = f(0) = f(1) \). Therefore \( f(x) = f(1) \) for all \( x \in \mathbb{R} \).

11. Suppose \( \lim_{x \to \infty} f(x) = \beta \) for some \( \beta \). Then there exists a positive real number \( M \) such that \( |f(x) - \beta| < 1 \) for all \( x \) such that \( x \geq M \). Then \( |f(x)| \leq 1 + \beta \) for every \( x \) such that \( x \geq M \). That is \( f \) is bounded on \( \{x: x \geq M\} \). Also by continuity, \( f \) is bounded on \([0, M]\).

12. Let \( f(x_n) \to f(x_0) \) for some \( x_n, x_0 \in [0, 1] \). We show that \( x_n \to x_0 \) which proves that \( f^{-1} \) is continuous at \( f(x_0) \). If \( (x_{n_k}) \) is any subsequence, then by Bolzano-Weierstrass theorem, there exists a subsequence \( (x_{n_{k_1}}) \) such that \( x_{n_{k_1}} \to \alpha \) for some \( \alpha \in [0, 1] \). By continuity, \( f(x_{n_{k_1}}) \to f(\alpha) \). By our assumption \( f(\alpha) = f(x_0) \) and since \( f \) is one-one \( x_0 = \alpha \). By Problem 8 of Practice problems 3, \( x_n \to x_0 \).

13. First observe that \( f(0) = 0 \) and \( f(n) = nf(1) \) for all \( n \in \mathbb{N} \). Next note that \( f(-1) = -f(1) \) and \( f(m) = f(1)m \) for all \( m \in \mathbb{Z} \). By observing \( f(\frac{1}{n}) = f(1)\frac{1}{n} \) for all \( n \in \mathbb{N} \), show that \( f(\frac{m}{n}) = f(1)\frac{m}{n} \) for all \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \). Finally take any irrational number \( x \) and find \( r_n \in \mathbb{Q} \) such that \( r_n \to x \) and apply the continuity to conclude that \( f(x) = f(1)x \).

14. (a) If for some \( M \in \mathbb{N}, q_n < M \) for all \( n \in \mathbb{N} \), then the set \( \{x_n : n \in \mathbb{N}\} \) is finite which is not true. Similarly we can show that any subsequence of \( (q_n) \) cannot be bounded.

(b) Suppose \( x_0 \) is rational in \((0, 1)\) and \( x_n \to x_0 \) where \( x_n \) can be rational or irrational. Apply (a) to show that \( f(x_n) \to 0 = f(x_0) \).

(c) Suppose \( x_0 \) is irrational in \((0, 1)\). To show that \( f \) is discontinuous at \( x_0 \), choose an irrational sequence \( (x_n) \) such that \( x_n \to x_0 \).