Practice Problems 5 : Existence of maxima/minima, Intermediate Value Property

1. Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(x) > 0$ for all $x \in [a, b]$. Show that there exists $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in [a, b]$.

2. Let $f : [0, 1] \to (0, 1)$ be an on-to function. Show that f is not continuous on $[0, 1]$.

3. Give an example of a function f on $[0, 1]$ which is not continuous but it satisfies the IVP (We say that f has the property IVP if for every $x, y \in [0, 1]$ and α satisfying $f(x) < \alpha < f(y)$ or $f(x) > \alpha > f(y)$ there exists $x_0 \in [x, y]$ such that $f(x_0) = \alpha$).

4. Show that the polynomial $x^4 + 6x^3 - 8$ has at least two real roots.

5. Show that there exists at least one positive real solution to the equation $|x^{31}+x^8+20| = x^{32}$.

6. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that there exists an $x_0 \in [0, 1]$ such that $f(x_0) = \frac{1}{3}(f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4}))$.

7. Let $f(x) = x^{2n} + a_{2n-1}x^{2n-1} + ... + a_1x + a_0$ where $n \in \mathbb{N}$ and $a_i's$ are in $\mathbb{R}$. Show that f attains its infimum on $\mathbb{R}$.

8. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ where $n > 0$ and $a_i's$ are in $\mathbb{R}$. If n is even and $a_n = 1$ and $a_0 = -1$, show that $f(x)$ has at least two real roots.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f is a constant function if
   (a) $f(x)$ is rational for each $x \in \mathbb{R}$.
   (b) $f(x)$ is an integer for each $x \in \mathbb{Q}$.

10. Let $f : [0, 1] \to R$. Suppose that $f(x)$ is rational for irrational x and that $f(x)$ is irrational for rational x. Show that f cannot be continuous.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.

12. Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that there exists $x_0 \in [0, \frac{1}{2}]$ such that $f(x_0) = f(x_0 + \frac{1}{2})$.

13. Let $f, g : [0, 1] \to \mathbb{R}$ be continuous such that $\inf\{f(x) : x \in [0, 1]\} = \inf\{g(x) : x \in [0, 1]\}$. Show that there exists $x_0 \in [0, 1]$ such that $f(x_0) = g(x_0)$.

14. A cross country runner runs continuously a eight kilometers course in 40 minutes without taking rest. Show that, somewhere along the course, the runner must have covered a distance of one kilometer in exactly 5 minutes.

15. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous one-one map. Show that f is either strictly increasing or strictly decreasing.

16. (*) Let $f : \mathbb{R} \to [0, \infty)$ be a bijective map. Show that f is not continuous on $\mathbb{R}$.

17. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.
    (a) Suppose f attains each of its values exactly two times. Let $f(x_1) = f(x_2) = \alpha$ for some $\alpha \in \mathbb{R}$ and $f(x) > \alpha$ for some $x \in [x_1, x_2]$. Show that f attains its maximum in $[x_1, x_2]$ exactly at one point.
    (b) Using (a) show that f cannot attain each of its values exactly two times.

Practice Problems 5: Hints/Solutions

1. Observe that f attains its minimum on $[a, b]$. Take $\alpha = inf\{f(x) : x \in [a, b]\}$.

2. The minimum value of f is 0 which is not attained by f.

3. Consider $f(0) = 0$ and $f(x) = \sin \frac{1}{x}$ for $x \neq 0$.

4. Note that $f(0) < 0, f(2) > 0$ and $f(-8) > 0$. Use IVP.

5. Let $f(x) = \frac{1}{x^{32}}|x^{31} + x^8 + 20| - 1$. Then $f(x) \to \infty$ as $x \to 0$ and $f(x) \to -1$ as $x \to \infty$. By IVP, there exists $x_0 \in (0, \infty)$ such that $f(x_0) = 0$.

6. Let $x_1, x_2 \in [0, 1]$ be such that $f(x_1) = inf\{f(x) : x \in [0, 1]\}$ and $f(x_2) = sup\{f(x) : x \in [0, 1]\}$. Note that $f(x_1) \leq \frac{1}{3}(f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4})) \leq f(x_2)$. Apply IVP.

7. Note that $f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. Let $M > 0$ be such that $M > f(0)$ or $f(y)$ for some $y \in \mathbb{R}$. Then there exists p such that $f(x) > M$ for all $|x| > p$. Since f is continuous there exists x_0 such that $f(x_0) = inf\{f(x) : x \in [-p, p]\} = inf\{f(x) : x \in \mathbb{R}\}$

8. Note that $f(0) = -1$ and $f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. Apply IVP.

9. (a) Suppose $f(x) \neq f(y)$ for some $x, y \in \mathbb{R}$. Find an irrational number α between $f(x)$ and $f(y)$. By IVP, there exists $z \in (x, y)$ such that $f(z) = \alpha$ which is a contradiction.

   (b) Let α be irrational. Find $r_n \in \mathbb{Q}$ such that $r_n \to \alpha$. By continuity $f(r_n) \to f(\alpha)$. Since $f(r_n)'s$ are integers, $(f(r_n))$ has to be eventually a constant sequence and hence $f(\alpha)$ is an integer. So f takes only integer value for each $x \in \mathbb{R}$. By IVP, $f(x)$ has to be constant.

10. Let g be defined by $g(x) = f(x) - x \; \forall \; x \in [0, 1]$. Then $g(x)$ irrational for all $x \in [0, 1]$. Because of IVP, g cannot be continuous and hence f cannot be continuous.

11. Consider the function $g(x) = f(x + \pi) - f(x)$ and the values $g(0)$ and $g(\pi)$. Apply IVP.

12. Consider the function $g(x) = f(x) - f(x + \frac{1}{2})$ and the values $g(0)$ and $g(\frac{1}{2})$. Apply IVP.

13. Let $x_1, x_2 \in [0, 1]$ be such that $f(x_1) = \inf\{f(x) : x \in [0, 1]\}$ and $g(x_2) = \inf\{g(x) : x \in [0, 1]\}$. Note that $f(x_1) \leq g(x_1)$ and $f(x_2) \geq g(x_2)$. Let $\varphi(x) = f(x) - g(x)$. Apply IVP to φ.

14. Let x denote the distance, in kilometers, along the course. Let $f : [0, 7] \to \mathbb{R}$, where $f(x) =$ time taken in minutes to cover the distance from x to $x+1$. Observe that $\sum_{i=0}^{7} f(i) = 40$. Hence $f(i) < 5$ or $f(i) > 5$ is not possible for all $i = 0$ to 7. Therefore, there exists $i, j \in [0, 7]$ such that $f(i) \leq 5 \leq f(j)$. By IVP there exists $c \in (i, j)$ such that $f(c) = 5$.

15. Suppose f is neither strictly increasing nor strictly decreasing. Then we can assume that there exists x_1, x_2 and x_3 such that $x_1 < x_2 < x_3$ and $f(x_1) > f(x_2)$ and $f(x_2) < f(x_3)$. Let α be such that $f(x_2) < \alpha < min\{f(x_1), f(x_3)\}$. By IVP, there exist $u_1 \in (x_1, x_2)$ and $u_2 \in (x_2, x_3)$ such that $f(u_1) = \alpha = f(u_2)$. Since f is one-one, $u_1 = u_2$ which is a contradiction.

16. If f is continuous, by Problem 15, f is either is strictly increasing or strictly decreasing. Suppose f is strictly increasing. Since f is on-to, there exists x_0 such that $f(x_0) = 0$. Then $f(x) < f(x_0)$ for all $x < x_0$ which is a contradiction.

17. (a) Let $\beta = max\{f(x) : x \in [x_1, x_2]\}$. If f attains β on $[x_1, x_2]$ at more than one point, then there exists $\gamma \in (\alpha, \beta)$ such that f attains γ more than twice which is a contradiction.

(b) Suppose f attains each of its values exactly two times. Let x_1, x_2, α and β be as in (a). Since f attains β exactly once in $[x_1, x_2]$, there exits x_0 lying outside $[x_1, x_2]$ such that $f(x_0) = \beta > \alpha$. Then, by IVP, every number in (α, β) is attained by f more than twice which is a contradiction.