Chapter 10

DOUBLE LEBESGUE INTEGRALS

10.1. Introduction

The purpose of this chapter is to extend the idea of the Lebesgue integral to functions of more than one real variable. Here we shall restrict ourselves to the case of functions of two real variables. The definitions and results here can be generalized in a very natural way to the case of functions of more than two real variables.

As in the one-variable case, Lebesgue integration for functions of two variables again is a generalization of Riemann integration. However, a new feature here is a result of Fubini which reduces the problem of calculating a two-dimensional integral to the problem of calculating one-dimensional integrals.

Again, our approach is via step functions and upper functions. Many of the details are similar to the one-variable case, and we shall omit some of the details.

We shall first of all make a few remarks on the problem of extending a number of definitions and results on point sets in \( \mathbb{R} \) to point sets in \( \mathbb{R}^2 \). The reader may wish to provide the proofs by generalizing those in Chapter 3.

Remarks. (1) We shall measure distance in \( \mathbb{R}^2 \) by euclidean distance; in other words, the distance between two points \( x, y \in \mathbb{R}^2 \) is given by \( |x - y| \), the modulus of the vector \( x - y \).

(2) We can define interior points in terms of open discs \( D(x, \epsilon) = \{ y \in \mathbb{R}^2 : |x - y| < \epsilon \} \) instead of open intervals \( (x - \epsilon, x + \epsilon) = \{ y \in \mathbb{R} : |x - y| < \epsilon \} \). Then open sets can be defined in the same way as before. Theorems 3A and 3B generalize easily.
(3) The limit of sequences in $\mathbb{R}^2$ can be defined in terms of the euclidean distance discussed in Remark (1). Then closed sets can be defined in the same way as before. Theorems 3D, 3E and 3F generalize easily.

(4) The Cantor intersection theorem in $\mathbb{R}^2$ can also be established via the Bolzano-Weierstrass theorem in $\mathbb{R}^2$, a simple consequence of the Bolzano-Weierstrass theorem in $\mathbb{R}$.

(5) An interval (resp. an open interval, a closed interval) in $\mathbb{R}^2$ is defined to be the cartesian product of two intervals (resp. open intervals, closed intervals) in $\mathbb{R}$. If $I$ is an interval in $\mathbb{R}^2$, then $\mu(I)$ denotes its area, the product of the lengths of the two intervals in $\mathbb{R}$ making up the cartesian product. Sets of measure zero and compact sets in $\mathbb{R}^2$ can be defined in the same way as before. Theorems 3L and 3M generalize easily. The Heine-Borel theorem can also be generalized: Any closed and bounded set in $\mathbb{R}^2$ is compact.

(6) As before, a property $P(x)$ is said to hold for almost all $x \in S$ if $P(x)$ fails to hold for at most a set of measure zero in $S$.

We next make a few remarks on the problem of extending a number of definitions and results on Riemann integration of functions of one variable to Riemann integration of functions of two variables. The reader may wish to provide the proofs by generalizing those in Chapter 2.

REMARKS. (1) Suppose that a function $f(x, y)$ is bounded on the interval $[A_1, B_1] \times [A_2, B_2]$, where $A_1, B_1, A_2, B_2 \in \mathbb{R}$ satisfy $A_1 < B_1$ and $A_2 < B_2$. Suppose further that

$$\Delta_1 : A_1 = x_0 < x_1 < x_2 < \cdots < x_n = B_1 \quad \text{and} \quad \Delta_2 : A_2 = y_0 < y_1 < y_2 < \cdots < y_m = B_2$$

are dissections of the intervals $[A_1, B_1]$ and $[A_2, B_2]$ respectively. We consider $\Delta = \Delta_1 \times \Delta_2$ to be a dissection of $[A_1, B_1] \times [A_2, B_2]$.

(2) The sums

$$s(f, \Delta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \inf_{x \in [x_{i-1}, x_i], y \in [y_{j-1}, y_j]} f(x, y)$$

and

$$S(f, \Delta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \sup_{x \in [x_{i-1}, x_i], y \in [y_{j-1}, y_j]} f(x, y)$$

are then the lower and upper Riemann sums respectively of $f(x, y)$ corresponding to the dissection $\Delta$.

(3) As before, we define the lower integral by taking the supremum of the lower sums over all dissections $\Delta$ of $[A_1, B_1] \times [A_2, B_2]$. Similarly, we define the upper integral by taking the infimum of the upper sums over all dissections $\Delta$ of $[A_1, B_1] \times [A_2, B_2]$. If the lower and upper integrals have the same value, then their common value is taken to be the Riemann integral

$$\int_{[A_1, B_1] \times [A_2, B_2]} f(x, y) \, d(x, y).$$

(4) All the results in Chapter 2 can be extended to the case of functions of two variables, and the proofs are similar but perhaps technically slightly more complicated in a few cases. Note also the very restrictive nature of the generalizations of Theorems 2F and 2G. Also, try to find the strongest generalization of Theorem 2H.
THEOREM 10A. The following result can be viewed as a generalization of Theorem 3C.

10.2. Decomposition into Squares

as before.

functions and all measurable functions on \( I \) over arbitrary measurable sets in \( \mathbb{R} \) is finite.

THEOREM 10B. Every open set \( G \subseteq \mathbb{R}^2 \) is measurable. Furthermore, if \( G \) is bounded, then \( \mu(G) \) is finite.
**THEOREM 10C.** Every closed set \( F \subseteq \mathbb{R}^2 \) is measurable. Furthermore, if \( F \) is bounded, then \( \mu(F) \) is finite.

**Proof of Theorem 10A.** For every \( m \in \mathbb{N} \) and \( k_1, k_2 \in \mathbb{Z} \), consider the square
\[
S(m, k_1, k_2) = \left[ \frac{k_1}{2^m}, \frac{k_1 + 1}{2^m} \right) \times \left[ \frac{k_2}{2^m}, \frac{k_2 + 1}{2^m} \right).
\]
with closure
\[
\overline{S}(m, k_1, k_2) = \left[ \frac{k_1}{2^m}, \frac{k_1 + 1}{2^m} \right] \times \left[ \frac{k_2}{2^m}, \frac{k_2 + 1}{2^m} \right].
\]
It is easy to see that for every \( m \in \mathbb{N} \), the collection
\[
Q_m = \{ S(m, k_1, k_2) : k_1, k_2 \in \mathbb{Z} \}
\]
is pairwise disjoint and countable. Suppose now that \( G \subseteq \mathbb{R}^2 \) is a given open set. Let
\[
S_1 = \bigcup_{S(m, k_1, k_2) \subseteq G} (1, k_1, k_2).
\]
For every \( m \in \mathbb{N} \) satisfying \( m \geq 2 \), let
\[
S_m = \left( \bigcup_{S(m, k_1, k_2) \subseteq G} S(m, k_1, k_2) \right) \setminus (S_1 \cup \ldots \cup S_{m-1}).
\]
Finally, let
\[
S = \bigcup_{m=1}^{\infty} S_m.
\]
Note that for each \( m \in \mathbb{N} \), the set \( S_m \) is the union of a countable number of squares in \( Q_m \). Also, the sets \( S_1, S_2, S_3, \ldots \) are pairwise disjoint. It follows from Theorem 1E that \( S \) is a countable union of disjoint squares whose closures are contained in \( G \). Clearly \( S \subseteq G \). To prove Theorem 10A, it suffices to prove that \( G \subseteq S \), so that \( G = S \). Suppose that \( (x, y) \in G \). Since \( G \) is open, it follows that there exists \( \epsilon > 0 \) such that
\[
(x - \epsilon, x + \epsilon) \times (y - \epsilon, y + \epsilon) \subseteq G.
\]
Now choose \( m \in \mathbb{N} \) so that \( 2^{-m} < \epsilon \). Then (the reader is advised to draw a picture) there exist \( k_1, k_2 \in \mathbb{Z} \) such that
\[
\frac{k_1}{2^m} \leq x < \frac{k_1 + 1}{2^m} \quad \text{and} \quad \frac{k_2}{2^m} \leq y < \frac{k_2 + 1}{2^m},
\]
so that \( (x, y) \in S(m, k_1, k_2) \). It is easy to see that
\[
\overline{S}(m, k_1, k_2) = \left[ \frac{k_1}{2^m}, \frac{k_1 + 1}{2^m} \right] \times \left[ \frac{k_2}{2^m}, \frac{k_2 + 1}{2^m} \right] \subseteq (x - \epsilon, x + \epsilon) \times (y - \epsilon, y + \epsilon) \subseteq G.
\]
In other words, there exist \( m \in \mathbb{N} \) and \( k_1, k_2 \in \mathbb{Z} \) such that
\[
(1) \quad (x, y) \in S(m, k_1, k_2) \quad \text{and} \quad \overline{S}(m, k_1, k_2) \subseteq G.
\]
Let \( m_0 \) be the smallest value of \( m \in \mathbb{N} \) such that there exist \( k_1, k_2 \in \mathbb{Z} \) for which (1) holds. It is easy to see that \( (x, y) \in S_{m_0} \subseteq S \). Hence \( G \subseteq S \). ☐
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Proof of Theorems 10B and 10C. Clearly each square is measurable. The first assertion of Theorem 10B follows from Theorem 10A and the two-dimensional analogue of Theorem 8H. To prove the first assertion of Theorem 10C, note that the set $G = \mathbb{R}^2 \setminus F$ is open and so measurable, so that $\chi_G \in \mathcal{M}(\mathbb{R}^2)$. But $\chi_F = 1 - \chi_G$. Hence $\chi_F \in \mathcal{M}(\mathbb{R}^2)$, whence $F$ is measurable. To complete the proof, note that a bounded measurable set $S$ is contained in a square of finite area $\mu(T)$. Clearly $\mu(S) \leq \mu(T)$.

\[ \square \]

10.3. Fubini’s Theorem for Step Functions

A useful result of Fubini reduces the problem of calculating a two-dimensional integral to the problem of calculating one-dimensional integrals. In this section and the next two, we shall establish this result. The special case for step functions is summarized by the following theorem.

THEOREM 10D. Suppose that $s \in \mathcal{S}(\mathbb{R}^2)$.

(a) For each fixed $y \in \mathbb{R}$, the integral $\int_{\mathbb{R}} s(x, y) \, dx$ exists and, as a function of $y$, is Lebesgue integrable on $\mathbb{R}$. Furthermore,

$$ \int_{\mathbb{R}^2} s(x, y) \, d(x, y) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} s(x, y) \, dx \right) \, dy. $$

(b) For each fixed $x \in \mathbb{R}$, the integral $\int_{\mathbb{R}} s(x, y) \, dy$ exists and, as a function of $x$, is Lebesgue integrable on $\mathbb{R}$. Furthermore,

$$ \int_{\mathbb{R}^2} s(x, y) \, d(x, y) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} s(x, y) \, dy \right) \, dx. $$

(c) In particular, we have

$$ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} s(x, y) \, dx \right) \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} s(x, y) \, dy \right) \, dx. $$

Proof. To prove (a), note that there exist $A_1, B_1, A_2, B_2 \in \mathbb{R}$ satisfying $A_1 < B_1$ and $A_2 < B_2$ such that $s : [A_1, B_1] \times [A_2, B_2] \to \mathbb{R}$ is a step function on $[A_1, B_1] \times [A_2, B_2]$ and $s(x, y) = 0$ for every $(x, y) \notin [A_1, B_1] \times [A_2, B_2]$. Hence there exist dissections $A_1 = x_0 < x_1 < x_2 < \ldots < x_n = B_1$ and $A_2 = y_0 < y_1 < y_2 < \ldots < y_m = B_2$ of $[A_1, B_1]$ and $[A_2, B_2]$ respectively, and numbers $c_{ij} \in \mathbb{R}$ such that for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we have $s(x, y) = c_{ij}$ for every $x \in (x_{i-1}, x_i)$ and $y \in (y_{j-1}, y_j)$. For every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the integral

$$ \int_{[x_{i-1}, x_i] \times [y_{j-1}, y_j]} s(x, y) \, d(x, y) = c_{ij} (x_i - x_{i-1})(y_j - y_{j-1}) = \int_{[y_{j-1}, y_j]} \left( \int_{[x_{i-1}, x_i]} s(x, y) \, dx \right) \, dy. $$

Summing over $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we obtain

$$ \int_{[A_1, B_1] \times [A_2, B_2]} s(x, y) \, d(x, y) = \int_{[A_1, B_1]} \left( \int_{[A_2, B_2]} s(x, y) \, dx \right) \, dy. $$

Since $s(x, y) = 0$ whenever $(x, y) \notin [A_1, B_1] \times [A_2, B_2]$, the result follows. Part (b) is similar. Part (c) follows immediately on combining (a) and (b). \[ \square \]
10.4. Sets of Measure Zero

The generalization of Theorem 10D to upper functions and Lebesgue integrable functions depends on the following result on sets of measure zero.

**Definition.** Suppose that \(S \subseteq \mathbb{R}^2\). For every \(y \in \mathbb{R}\), we write \(S_1(y) = \{x \in \mathbb{R} : (x, y) \in S\}\). For every \(x \in \mathbb{R}\), we write \(S_2(x) = \{y \in \mathbb{R} : (x, y) \in S\}\).

**Theorem 10E.** Suppose that \(S \subseteq \mathbb{R}^2\), and that \(\mu(S) = 0\). Then
(a) \(\mu(S_1(y)) = 0\) for almost all \(y \in \mathbb{R}\); and
(b) \(\mu(S_2(x)) = 0\) for almost all \(x \in \mathbb{R}\).

The proof of Theorem 10E depends on the following equivalent formulation for sets of measure zero.

**Theorem 10F.**
(a) Suppose that \(S \subseteq \mathbb{R}\). Then \(\mu(S) = 0\) if and only if there exists a sequence of intervals \(I_n \in \mathbb{R}\) such that
\[
\sum_{n=1}^{\infty} \mu(I_n) < \infty \text{ and every } x \in S \text{ belongs to infinitely many } I_n.
\]
(b) Suppose that \(S \subseteq \mathbb{R}^2\). Then \(\mu(S) = 0\) if and only if there exists a sequence of intervals \(J_n \in \mathbb{R}^2\) such that
\[
\sum_{n=1}^{\infty} \mu(J_n) < \infty \text{ and every } (x, y) \in S \text{ belongs to infinitely many } J_n.
\]

**Proof.** The proofs of the two parts are similar, so we shall only prove (a).

(\(\Rightarrow\)) For every \(m \in \mathbb{N}\), there exists a sequence of intervals \(I_n^{(m)} \subseteq \mathbb{R}\) such that
\[
S \subseteq \bigcup_{n=1}^{\infty} I_n^{(m)} \quad \text{and} \quad \sum_{n=1}^{\infty} \mu(I_n^{(m)}) < 2^{-m}.
\]
Then the collection \(Q = \{I_n^{(m)} : m, n \in \mathbb{N}\}\) is countable and
\[
\sum_{I \in Q} \mu(I) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(I_n^{(m)}) < 1.
\]
Clearly every \(x \in S\) belongs to infinitely many intervals in \(Q\).

(\(\Leftarrow\)) Given any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that
\[
\sum_{n=N}^{\infty} \mu(I_n) < \epsilon.
\]
Since every \(x \in S\) belongs to infinitely many \(I_n\), it follows that
\[
x \in \bigcup_{n=N}^{\infty} I_n.
\]
Hence

\[ S \subseteq \bigcup_{n=N}^{\infty} I_n \quad \text{and} \quad \sum_{n=N}^{\infty} \mu(I_n) < \epsilon. \]

The result follows. \( \Box \)

**Proof of Theorem 10E.** We shall only prove (a), since (b) is similar. Since \( \mu(S) = 0 \), it follows from Theorem 10F(b) that there exists a sequence of intervals \( J_n \in \mathbb{R}^2 \) such that

\[ \sum_{n=1}^{\infty} \mu(J_n) \]

is finite and every \((x, y) \in S\) belongs to infinitely many \( J_n \). For every \( n \in \mathbb{N} \), write \( J_n = X_n \times Y_n \), where the intervals \( X_n, Y_n \subseteq \mathbb{R} \). Then (note that we slightly abuse notation and use \( \mu \) to denote measure both in \( \mathbb{R} \) and in \( \mathbb{R}^2 \))

\[ \mu(J_n) = \mu(X_n) \mu(Y_n) = \mu(X_n) \int_{\mathbb{R}} \chi_{Y_n}(y) \, dy, \]

where \( \chi_{Y_n} \) denotes the characteristic function of \( Y_n \). Consider now the function \( g_n : \mathbb{R} \to \mathbb{R} \), defined by \( g_n(y) = \mu(X_n) \chi_{Y_n}(y) \) for every \( y \in \mathbb{R} \). Clearly \( g_n \in \mathcal{L}(\mathbb{R}) \), is non-negative, and

\[ \sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n(y) \, dy \]

converges. It follows from the Monotone convergence theorem (Theorem 5D) that

\[ \sum_{n=1}^{\infty} g_n(y) \]

converges for almost all \( y \in \mathbb{R} \). In other words,

\[ \sum_{n=1}^{\infty} \mu(X_n) \chi_{Y_n}(y) \]

converges for almost all \( y \in \mathbb{R} \). Suppose now that \( y \in \mathbb{R} \) and (2) converges. We shall show that \( \mu(S_1(y)) = 0 \). We may assume, without loss of generality, that \( S_1(y) \neq \emptyset \). Clearly

\[ Q(y) = \{ X_n : n \in \mathbb{N} \text{ and } y \in Y_n \} \]

is a countable collection of intervals in \( \mathbb{R} \), of total length (2). Furthermore, if \( x \in S_1(y) \), then \((x, y) \in S\), so that \((x, y)\) belongs to infinitely many \( J_n \), whence \( x \) belongs to infinitely \( X_n \) in \( Q(y) \). The result now follows from Theorem 10F(a). \( \Box \)

### 10.5. Fubini’s Theorem for Lebesgue Integrable Functions

We complete this chapter by proving the following result.
THEOREM 10G. Suppose that \( f \in \mathcal{L}(\mathbb{R}^2) \). Then

(a) the Lebesgue integral \( \int_{\mathbb{R}} f(x, y) \, dx \) exists for almost all \( y \in \mathbb{R} \), the function \( G : \mathbb{R} \to \mathbb{R} \), defined by

\[
G(y) = \begin{cases} 
\int_{\mathbb{R}} f(x, y) \, dx & \text{if the integral exists,} \\
0 & \text{otherwise,}
\end{cases}
\]

is Lebesgue integrable on \( \mathbb{R} \), and

\[
\int_{\mathbb{R}^2} f(x, y) \, d(x, y) = \int_{\mathbb{R}} G(y) \, dy;
\]

(b) the Lebesgue integral \( \int_{\mathbb{R}} f(x, y) \, dy \) exists for almost all \( x \in \mathbb{R} \), the function \( H : \mathbb{R} \to \mathbb{R} \), defined by

\[
H(x) = \begin{cases} 
\int_{\mathbb{R}} f(x, y) \, dy & \text{if the integral exists,} \\
0 & \text{otherwise,}
\end{cases}
\]

is Lebesgue integrable on \( \mathbb{R} \), and

\[
\int_{\mathbb{R}^2} f(x, y) \, d(x, y) = \int_{\mathbb{R}} H(x) \, dy;
\]

and

(c) we have

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) \, dx \right) \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) \, dy \right) \, dx.
\]

PROOF. If \( f \in \mathcal{S}(\mathbb{R}^2) \), then the result is given by Theorem 10D. To prove Theorem 10G, we shall first consider the special case when \( f \in \mathcal{U}(\mathbb{R}^2) \). If \( f \in \mathcal{U}(\mathbb{R}^2) \), then there exists an increasing sequence of step functions \( s_n \in \mathcal{S}(\mathbb{R}^2) \) such that \( s_n(x, y) \to f(x, y) \) as \( n \to \infty \) for all \( (x, y) \in \mathbb{R}^2 \setminus S \), where \( \mu(S) = 0 \), and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} s_n(x, y) \, d(x, y) = \int_{\mathbb{R}^2} f(x, y) \, d(x, y).
\]

Note that \((x, y) \in S \) if and only if \( x \in S_1(y) \), and that \( \mu(S_1(y)) = 0 \) in view of Theorem 10E. It follows that for every fixed \( y \in \mathbb{R} \), \( s_n(x, y) \to f(x, y) \) as \( n \to \infty \) for all \( x \in \mathbb{R} \setminus S_1(y) \). Note that by Theorem 10D, the integral

\[
t_n(y) = \int_{\mathbb{R}} s_n(x, y) \, dx
\]

exists for every \( y \in \mathbb{R} \) and, as a function of \( y \), is integrable on \( \mathbb{R} \). Furthermore,

\[
\int_{\mathbb{R}} t_n(y) \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} s_n(x, y) \, dx \right) \, dy = \int_{\mathbb{R}^2} s_n(x, y) \, d(x, y) \leq \int_{\mathbb{R}^2} f(x, y) \, d(x, y).
\]

It is easy to see that \( t_n \) is an increasing sequence on \( \mathbb{R} \), so that the left hand side of (4) is increasing and bounded above, and so converges. It follows from the Monotone convergence theorem (Theorem 5C) that there exists a function \( t \in \mathcal{L}(\mathbb{R}) \) such that \( t_n(y) \to t(y) \) as \( n \to \infty \) for all \( y \in \mathbb{R} \setminus T \), where \( \mu(T) = 0 \), and

\[
\int_{\mathbb{R}} t(y) \, dy = \lim_{n \to \infty} \int_{\mathbb{R}} t_n(y) \, dy.
\]
We also have
\[ t_n(y) = \int_{\mathbb{R}} s_n(x, y) \, dx \leq t(y) \]
for all \( y \in \mathbb{R} \setminus T \). For any \( y \in \mathbb{R} \setminus T \), it follows again from the Monotone convergence theorem (Theorem 5C) that there exists a function \( g \in \mathcal{L}(\mathbb{R}) \) such that \( s_n(x, y) \to g(x, y) \) as \( n \to \infty \) for almost all \( x \in \mathbb{R} \setminus W_y \), where \( \mu(W_y) = 0 \), and
\[ \int_{\mathbb{R}} g(x, y) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}} s_n(x, y) \, dx. \]
This means that
\[ f(x, y) = g(x, y) \quad \text{for all } y \in \mathbb{R} \setminus T \text{ and } x \in \mathbb{R} \setminus (S_1(y) \cup W_y). \]
It follows that for all \( y \in \mathbb{R} \setminus T \), the integral
\[ \int_{\mathbb{R}} f(x, y) \, dx \]
exists, and
\[ \int_{\mathbb{R}} f(x, y) \, dx = \int_{\mathbb{R}} g(x, y) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}} s_n(x, y) \, dx = t(y). \]
Since \( t \in \mathcal{L}(\mathbb{R}) \), it follows that \( G(y) \) is Lebesgue integrable on \( \mathbb{R} \). Combining (3)–(5), we obtain
\[ \int_{\mathbb{R}} t(y) \, dy = \int_{\mathbb{R}^2} f(x, y) \, d(x, y). \]
Also, (6) gives
\[ \int_{\mathbb{R}} t(y) \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) \, dx \right) \, dy. \]
We can now combine (7) and (8) to complete the proof of (a) when \( f \in \mathcal{U}(\mathbb{R}^2) \). Suppose now that \( f \in \mathcal{L}(\mathbb{R}^2) \). Then \( f = u - v \), where \( u, v \in \mathcal{U}(\mathbb{R}^2) \). Hence
\[ \int_{\mathbb{R}^2} f(x, y) \, d(x, y) = \int_{\mathbb{R}^2} u(x, y) \, d(x, y) - \int_{\mathbb{R}^2} v(x, y) \, d(x, y) \]
\[ = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} u(x, y) \, dx \right) \, dy - \int_{\mathbb{R}} \left( \int_{\mathbb{R}} v(x, y) \, dx \right) \, dy \]
\[ = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (u(x, y) - v(x, y)) \, dx \right) \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) \, dx \right) \, dy. \]
This completes the proof of (a). Part (b) is similar. Part (c) is a simple consequence of (a) and (b). \( \Box \)