# MTH102N
# ASSIGNMENT–LA 6

(1) Describe all $2 \times 2$ orthogonal matrices. Prove that action of any orthogonal matrix on a vector $v \in \mathbb{R}^2$, is either a rotation or a reflection about a line.

(2) Let $v, w \in \mathbb{R}^n$, $n \geq 2$, with $\|v\| = \|w\| = 1$. Prove that there exist an orthogonal matrix $A$ such that $A(v) = w$. Prove also that $A$ can be chosen such that $\det(A) = 1$.

(*This is why orthogonal matrices with determinant one are called rotations*))

(3) Let $a$ be a $m \times n$ matrix, that is, as a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. Let $N(A)$ =Kernel of $A$, $C(A)$ =Column space of $A$ and $R(A)$ =Row space of $A$. Prove that:

$$i)\ N(A) \perp R(A), \qquad ii)\ N(A) \oplus R(A) = \mathbb{R}^n.$$

(4) Let $A$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that $\det(A) = \lambda_1 \dots \lambda_n$ and $\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$. Further show that $A$ is invertible if and only if its all eigenvalues are non-zero.

(5) Let $A$ be an $n \times n$ invertible matrix. Show that eigenvalues of $A^{-1}$ are reciprocal of the eigenvalues of $A$ and $A$ and $A^{-1}$ have the same eigenvectors.

(6) Let $A$ be an $n \times n$ matrix and $\alpha$ be a scalar. Find the eigenvalues of $A - \alpha I$ in terms of eigenvalues of $A$. Further show that $A$ and $A - \alpha I$ have the same eigenvectors.

(7) Let $A$ be an $n \times n$ matrix. Show that $A^t$ and $A$ have the same eigenvalues. Do they have the same eigenvectors?

(8) Let $A$ be an $n \times n$ matrix. Show that:
  (a) If $A$ is idempotent ($A^2 = A$) then eigenvalues of $A$ are either 0 or 1.
  (b) If $A$ is nilpotent ($A^m = 0$ for some $m \geq 1$) then all eigenvalues of $A$ are 0.

(9) Find the eigenvalues and corresponding eigenvectors of matrices
(a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$

(10) Construct a basis of $\mathbb{R}^3$ consisting of eigenvectors of the following matrices
(a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

(11) Show that $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$ is diagonalizable. Find a matrix $Q$ such that $Q^{-1}AQ$ is a diagonal matrix.

(12) Let $A = \begin{pmatrix} 7 & -5 & 15 \\ 6 & -4 & 15 \\ 0 & 0 & 1 \end{pmatrix}$. Find a matrix $Q$ such that $Q^{-1}AQ$ is a diagonal matrix and hence calculate $A^6$.