MTH102N
ASSIGNMENT–C1

1. For any $z \in \mathbb{C}$, show that

   (a) Re $(iz) = -$Im z .

   (b) z is a ral number if and only if $z = \overline{z}$

   (c) $|\text{Re } z| \leq |z|$ and $|\text{Im } z| \leq |z|$.

   (d) $|\text{Im }(1 - \overline{z} + z^2)| < 3 \;\; \forall z < 1$.

2. Prove the following:

   (a) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$.

   (b) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

   (c) $|z_1 + z_2| \leq |z_1| + |z_2|$ and equality holds if and only if one is a nonnegative scalar multiple of the other.

3. Show that the equation $z^4+z+5 = 0$ has no solution in the set $\{z \in \mathbb{C} : \; |z| < 1\}$.

4. Let $\lambda \in \mathbb{C}$ be such that $0 < |\lambda| < 1$. Then show that

   (a) $|z - \lambda| < |1 - \overline{\lambda}z|$ if $|z| < 1$.

   (b) $|z - \lambda| = |1 - \overline{\lambda}z|$ if $|z| = 1$.

   (c) $|z - \lambda| > |1 - \overline{\lambda}z|$ if $|z| > 1$.

5. Sketch each of the following set of complex numbers.

   (a) $S = \{z : \; |z - 2 + i| \leq 1\}$.

   (b) $S = \{z : |2z + 3| > 4\}$

   (c) $S = \{z : \; |z - 1| = |z - 3|\}$.

   (d) $S = \{z : \; 1 < |z| < 2, \text{ Re } z \neq 0\}$

6. If z and w are in $\mathbb{C}$ such that Im$(z) > 0$ and Im$(w) > 0$, show that $|\frac{z-w}{z-\overline{w}}| < 1$.

7. Let $z = \frac{i}{-2-2i}$

   (a) Express z in polar form.

   (b) Express z^5 in polar and cartesian form.

   (c) Express $z^{1/5}$ in Cartesian form.

8. Prove that for $z, w \in \mathbb{C}$

$$|1 - z\overline{w}|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2).$$

   Using above deduce that if $|w| < 1$ then the function $f_w(z) = \frac{z-w}{1-z\overline{w}}$ maps the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ onto itself and the unit circle $S = \{z \in \mathbb{C} : |z| = 1\}$ onto itself.

9. Prove de Moivre's theorem: Given $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Use this result to find

$$a)\ (1 + i\sqrt{3})^{99} \qquad b)\ (\frac{1+i}{\sqrt{2}})^{10}.$$

10. Show that $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$ for any complex number z except $z = 1$, and deduce that

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}},$$

$0 < \theta < 2\pi$.

11. Discuss the convergence of the following sequences:

$$a)\ \{\cos(\frac{n\pi}{2}) + i^n\}, \qquad b)\ \{i^n \sin(\frac{n\pi}{4})\}, \qquad c)\ \{\frac{1}{n} + i^n\}.$$

12. Let $z = re^{i\theta}$, $w = Re^{i\phi}$, $0 \leq r < R$. For a fixed w find

$$\lim_{r \to R} \mathrm{Re}(\frac{w+z}{w-z}).$$

13. If $1 = z_0, z_1, \ldots, z_{n-1}$ are distinct n^{th} roots of unity, prove that $\Pi_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j$.

14. Check if the following functions can be prescribed a value at $z = 0$, so that they become continuous: $f(z) = \frac{|z|^2}{z}$, $f(z) = \frac{z+1}{|z|-1}$, $f(z) = \frac{\bar{z}}{z}$.