First Midsemester Examination

MTH102N

Second Semester, 2008-09

Time: 1 Hour **Marks: 40**

Notation: $B_1(0) = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{B_1(0)} = \{z \in \mathbb{C} : |z| \leq 1\}$.

1.a) Let γ be a circle in $\mathbb{C}$ oriented counter clockwise. If the center of γ is at zero prove that

$$\int_\gamma |z| dz = 0.$$

b) Let $\{z_n\} \subset B_1(0)$ be a sequence such that $z_n \neq 0$, for all n and $z_n \to 0$. Does there exist any nonconstant function f which is analytic on $B_1(0)$ and $f(z_n) = (-1)^n z_n$? Justify your answer. [5+5]

2.a) Evaluate the integral $\int_\gamma \frac{1}{(z-3)(z^2-1)} dz$, where $\gamma(t) = 2e^{it}$, $t \in [0, 2\pi)$.

b) Let P be a polynomial of degree $n \geq 1$ with distinct roots. Let γ be a simple closed curve oriented counter clockwise, which does not pass through any root of P but encloses all roots of P. If P' denotes the derivative of P then find the value of the integral

$$\int_\gamma \frac{P'(z)}{P(z)} dz.$$

[5+5]

3.a) Let α and β be complex numbers such that $0 < |\alpha| < |\beta|$. Find the Laurent series of the function

$$f(z) = \frac{1}{(z-\alpha)(z-\beta)},$$

in the annuli $A = \{z \in \mathbb{C} : |\alpha| < |z| < |\beta|\}$ and $B = \{z \in \mathbb{C} : |z| > |\beta|\}$.

b) Find all entire functions f which satisfies the inequality $|f(z)| \leq 1 + |z|^{\frac{1}{2}}$, for all $z \in \mathbb{C}$. [5+5]

4.a) Let $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be a function such that $\text{Re}f$ and $\text{Im}f$ has continuous partial derivatives on $\mathbb{C}$. If $g(z) = [f(z)]^2$ is an analytic function on $\mathbb{C}$ then prove that f is analytic on $\mathbb{C}$.

b) For $i = 1, 2$ let $f_i : \overline{B_1(0)} \to \mathbb{C} \setminus \{0\}$ be analytic functions. If $|f_1(z)| = |f_2(z)|$ for all z with $|z| = 1$ then prove that there exists a $\theta \in [0, 2\pi)$, independent of z, such that $f_1(z) = e^{i\theta} f_2(z)$ for all $z \in \overline{B_1(0)}$. [5+5]