Midsem I: Tentative marking scheme

1. a) Let $\gamma(t) = \rho e^{it}$, $t \in [0, 2\pi)$ for some $\rho > 0$. — ①

Then $\int_\gamma |z|\,dz = \int_0^{2\pi} i\rho^2 e^{it}\,dt$ — ②

$= i\rho^2 \frac{e^{it}}{i}\Big|_0^{2\pi} = 0$ — ②

b) Suppose $\exists$ an analytic fn. f satisfying the above.
Then $f(z_{2n}) = z_{2n}$ $\forall n$, $f(z_{2n+1}) = -z_{2n+1}$ — ①
Since $\{z_n\} \to 0$ we have $\{z_{2n}\} \to 0$, $\{z_{2n+1}\} \to 0$ — ①
Since $0 \in B_1(0)$ by identity theorem $f(z) = z$ & $f(z) = -z$ — ②
This is a contradiction so no such f exist.

2. a) $$\int_\gamma f(z)\,dz = \int_\gamma \frac{dz}{(z-3)(z^2-1)} = -\frac{1}{2}\left[\int_\gamma \frac{dz}{(z-3)(z+1)} - \int_\gamma \frac{dz}{(z-3)(z-1)}\right]$$ — ①

$$= \frac{2\pi i}{-2}\left(-\frac{1}{4} + \frac{1}{2}\right)$$

as $\gamma(t) = 2e^{it}$ includes the points $-1, 1$. — ③

$\therefore \int_\gamma f(z)\,dz = -\frac{i\pi}{4}$ — ①

b) Let $z_1, z_2, \dots, z_n$ be zeros of P and $P(z) = \alpha(z-z_1)\cdots(z-z_n)$ — ②
where $\alpha \in \mathbb{C}$. Then $\frac{P'(z)}{P(z)} = \sum_{i=1}^{n} \frac{1}{z-z_i}$ — ②

$\Rightarrow$ By Cauchy's integral formula

$$\int_\gamma \frac{P'(z)}{P(z)}\,dz = \sum_{i=1}^{n} \int_\gamma \frac{dz}{z-z_i} = 2\pi i n$$ — ①

3. a) On A_1: $f(z) = \frac{1}{\alpha-\beta}\left(\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right)$

$$= \frac{1}{\alpha-\beta}\left(\frac{1}{z\left(1-\frac{\alpha}{z}\right)} + \frac{1}{\beta\left(1-z/\beta\right)}\right)$$ — ②

$$= \sum_{n=0}^{\infty}\left(\frac{\alpha^n}{\alpha-\beta}\right)\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{\beta^{n+1}(\alpha-\beta)}z^n$$ — ①

On A_2: $f(z) = \frac{1}{\alpha-\beta}\left(\frac{1}{z\left(1-\frac{\alpha}{z}\right)} - \frac{1}{z\left(1-\beta/z\right)}\right)$ — ①

$$= \sum_{n=0}^{\infty}\left(\frac{\alpha^n - \beta^n}{\alpha-\beta}\right)\frac{1}{z^{n+1}}$$ — ①

b) By Cauchy's estimate we have for all $K \geq 1$

$$|f^{K}(0)| \leq \frac{K!}{R^K} \sup_{|z|=R} |f(z)| \leq \frac{K!}{R^K}(1+R^{1/2}) \quad — ②$$

$$\to 0 \text{ as } R \to \infty$$

$\Rightarrow f(z) = \alpha$ (a Const.) — ③

$\Rightarrow |\alpha| = |f(0)| \leq 1$ — ①

4. a) Let $g(x,y) = u^2(x,y) - v^2(x,y) + 2i\,u(x,y)\,v(x,y)$ — ①

Since g is analytic we have from CR eqn.

$$\left.\begin{aligned} 2uu_x - 2vv_x &= 2uv_y + 2u_y v \quad \cdots (i) \\ -2uv_x - 2u_x v &= 2uu_y - 2vv_y \quad \cdots (ii) \end{aligned}\right\} — ②$$

$$\left.\begin{aligned} (i)\times v + (ii)\times u &\Rightarrow -2(u^2+v^2)v_x = 2(u^2+v^2)u_y \\ (i)\times u - (ii)\times v &\Rightarrow 2(u^2+v^2)u_x = 2(u^2+v^2)v_y \end{aligned}\right\} — ②$$

$f(z) \neq 0 \;\forall z \Rightarrow |f(z)|^2 = u^2+v^2 \neq 0 \Rightarrow u_x = v_y,\; u_y = -v_x$.

Since u, v, u_x, v_x, u_y, v_y are cont. f is analytic.

b) Let $h(z) = \dfrac{f(z)}{g(z)}$, — ①

then h is analytic on $\overline{B_1(0)}$.

Since $|h(z)| = 1 \;\forall z$ with $|z| = 1$, by maximum modulus principle $|h(z)| \leq 1 \;\forall z \in \overline{B_1(0)}$. — ①

Similarly $|h(z)| \geq 1 \;\forall z \in \overline{B_1(0)}$ by considering $\dfrac{1}{h(z)} = \dfrac{g(z)}{f(z)}$.

$\Rightarrow |h(z)| = 1 \;\forall z \in \overline{B_1(0)}$ — ①

Since h is analytic it follows from CR eqn. that $h(z) = \alpha \;\forall z \in \overline{B_1(0)}$. — ①

As $|h(z)| = 1$, $\alpha = e^{i\theta}$ for $\theta \in [0, 2\pi)$. — ①