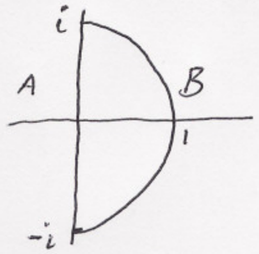
Tentative marking scheme:

1. a) The given set is



$$\left.\begin{aligned} f(0) &= 1 \\ f(i) &= i \\ f(-i) &= -i \\ f(1) &= \infty \\ f\left(\tfrac{1}{2}\right) &= 3 \end{aligned}\right\} - ②$$

So image of A is B and Image of B is the imaginary axis. By deleting the boundary and using connectedness argument it follows that $\operatorname{Im} f = \{ z \in \mathbb{C} / \operatorname{Re} z \geq 0, |z| \geq 1 \}$ — ③

b) $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$ — ①

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2(2n+1)!} \quad — ②$$

$$\Rightarrow \operatorname{Res}_{z=0} \frac{\sinh z}{z^3} = 0 \quad — ②$$

2. a) Using $z = e^{i\theta}$ — ①

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \int_{|z|=1} \frac{dz}{iz\left(1 + \frac{a}{2}\left(z + \frac{1}{z}\right)\right)} \quad — ①$$

$$= \frac{2}{i}\int_{|z|=1} \frac{dz}{az^2 + 2z + a} = \frac{2}{i}\int_{|z|=1} \frac{dz}{\left(z - \left(\frac{-1}{a} + \frac{\sqrt{1-a^2}}{a}\right)\right)\left(z - \left(\frac{-1}{a} - \frac{\sqrt{1-a^2}}{a}\right)\right)} \quad — ①$$

$$= \frac{2\pi a}{\sqrt{1-a^2}} \text{ as } \left| -\frac{1}{a} - \frac{\sqrt{1-a^2}}{a} \right| > 1 \quad — ②$$

b) AB skew symmetric $\underset{①}{\Rightarrow} (AB)^t = -AB \underset{①}{\Rightarrow} B^t A^t = -AB \underset{①}{\Rightarrow} BA = -AB$

(For converse give ② upto $(AB)^t = -AB$).

3. a) $\operatorname{Tr}(A-B) = \sum_{i=1}^{n} (a_{ii} - b_{ii})$ — ①

$$= \sum_{i=1}^{n} a_{ii} - \sum_{i=1}^{n} b_{ii}$$

$$= \operatorname{Tr}(A) - \operatorname{Tr}(B) \quad — ①$$

If $AB - BA = I$ then $\operatorname{Tr}(AB - BA) = n$ — ②

$\Rightarrow Tr(AB) - Tr(BA) = n$

$\Rightarrow 0 = n$ as $Tr(AB) = Tr(BA)$ — ①

b) clearly $\left|\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right| = |A|$, $\left|\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right| = |B|$ — ②

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \text{ — ②}$$

$$\Rightarrow \left|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right| = \left|\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right|\left|\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right| = |A||B| \text{ — ①}$$

4. a) $|A| = 1 \Rightarrow A \cdot Adj A = I \Rightarrow |Adj A| = 1$ — ②

$A^{-1} = Adj A \Rightarrow (Adj(A))^{-1} = A$ — ①

$\Rightarrow \frac{1}{|Adj A|} Adj(Adj A) = Adj(Adj A) = A$ — ②

b) Suppose that A is not invertible. Then $EAF = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ where $r < n$ & E, F are product of elementary matrices. — ②

$\Rightarrow AF = E^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ has last column consisting of zeros only. — ①

$\Rightarrow$ if F_n is the last column of F then $AF_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ — ①

As F is invertible last column of F cannot consist of zeros only - contradiction. — ①
