Tentative marking scheme for end sem.

1.a) True. — ①

If $A = SDS^{-1}$ then diagonal elements of D are eigen values of A. — ①

Since $A^7 = 0$ only eigen value of A is zero. As $A = SDS^{-1} \Rightarrow A = 0$ — ①

b) False. — ①

$|A - \lambda I| = (\lambda - 2)^3 \Rightarrow \lambda = 2.$ — ①

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \Rightarrow E(2) = \{(x,0,0) / x \in \mathbb{R}\}$$

$\Rightarrow \nexists$ no basis consisting of eigenvectors of A. — ①

c) True. — ① As $T \neq 0$ and $T: V \to \mathbb{R} \Rightarrow$ rank$(T) = 1$ — ①

$\Rightarrow$ Nullity$(T) = 3$ as $\dim(V) = 4$. — ①

d) False. — ① As $T(0,0,0) \neq (0,0,0)$. T is not linear. — ②

e) True. — ①

As A is product of elementary matrices A is invertible. — ①

$\therefore x = A^{-1}b$. — ①

f) True. — ①

T is onto $\Rightarrow$ rank$(T) = 3$ — ①

$\Rightarrow \ker(T) = 0 \Rightarrow T$ is 1-1. — ①

2.a) $V \cap W = \{(v, w, x, y, z) : y + x - 3v + z = 0,\ z + x - w = 0,\ v = y\}$

$\Rightarrow z = 2y - x$ (from 1st eqn. & $v = y$)

$w = z + x = 2y$ (from 2nd eqn.)

$\therefore V \cap W = \{(y, 2y, x, y, 2y - x) / x \in \mathbb{R}, y \in \mathbb{R}\}$ — ②

$= \text{span}\{(0,0,1,0,-1), (1,2,0,1,2)\}$

$\therefore \{(0,0,1,0,-1), (1,2,0,1,2)\}$ is a basis of $W \cap V$ — ①

$\dim W \cap V = 2$.

As $W = \{(y, x+z, x, y, z) / x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$, $(1,1,1,1,0) \notin W \cap V$

but $(1,1,1,1,0) \in W$. — ③

As $W = \text{Span}\{(0,1,1,0,0), (1,0,0,1,0), (0,1,0,0,1)\}$, $\dim W = 3$ — ②

$\Rightarrow \{(0,0,1,0,-1), (1,2,0,1,2), (1,1,1,1,0)\}$ is a basis of W. — ①

b) Nullity $\geq 1 \Rightarrow$ rank $\leq 3 \Leftrightarrow |A| = 0$ — ③

$$\begin{vmatrix} m & -1 & 0 & 0 \\ 0 & m & -1 & 0 \\ 0 & 0 & m & -1 \\ -6 & 11 & -6 & 1 \end{vmatrix} = (m-1)(m-2)(m-3) \text{ — ②}$$

$\therefore$ For $m = 1, 2, 3$ rank$(A) \leq 3$.

3. a) $T(1) = x^2$, $T(x-1) = 0$, $T((x-1)^2) = x - x^3$, $T((x-1)^3) = 1 - 2x + 2x^2$ — ④

Hence matrix of $T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 2 \end{pmatrix}$ — ②

coordinate of $T(8+4(x-1)+2(x-1)^2+(x-1)^3)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix} \text{ — ②}$$

b) As $(3,1,1,-2,3) = (1,1,1,0,1) + 2(1,0,0,-1,1)$, $W = Sp\{(1,1,1,0,1),(1,0,0,-1,1)\}$ — ①

Let $u_1 = (1,1,1,0,1)$, $u_2 = (1,0,0,-1,1)$. Then by G.S. orthogonalization

$w_1 = (1,1,1,0,1)$ — ①

$w_2 = u_2 - \dfrac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$ — ②

$= (1,0,0,-1,1) - \frac{2}{4}(1,1,1,0,1) = (1/2, -1/2, -1/2, -1, 1/2)$ — ①

By normalizing w_1 and w_2 we get

$\{(1/2, 1/2, 1/2, 0, 1/2), (1/2\sqrt{2}, -1/2\sqrt{2}, -1/2\sqrt{2}, -1/\sqrt{2}, 1/2\sqrt{2})\}$. — ①

Note: If one does not notice the first line then he/she will find $w_3 = 0$ — ①.

4. a) $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda - 5)(\lambda + 1)$

$\therefore \lambda = 5$, $\lambda = -1$ are eigen values. — ①

For $\lambda = -1$, $\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow E(-1) = \{(x,y) / x = -2y\} = Sp\{(-2,1)\}$. — ①

For $\lambda = 5$, $\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow E(5) = \{(x,y) / x = y\} = Sp\{(1,1)\}$ — ①

Since $(-2,1)$ and $(1,1)$ are linearly independent A is diagonalizable and $A = QDQ^{-1}$, — ①

with $Q = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$ — ①. $Q^{-1} = \dfrac{Adj\, Q}{|Q|} = \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$ — ①

①

$\therefore$ $B = A^{100} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$ — ①

b) Since $\{(0,0,1), (1/\sqrt{2}, -1/\sqrt{2}, 0)\}$ is an orthonormal basis of W — ①

the point in W closest to $(0,1,1)$ is

$\langle (0,0,1),(0,1,1)\rangle (0,0,1) + \langle (1/\sqrt{2}, -1/\sqrt{2}, 0),(0,1,1)\rangle (1/\sqrt{2}, -1/\sqrt{2}, 0)$ — ①

$= (0,0,1) + (-1/2, 1/2, 0) = (-1/2, 1/2, 1)$ — ①

c) Range of $A = Sp\{Ae_1, \ldots, Ae_n\}$ — ①

$= Sp\{c_1, c_2, \ldots, c_n\}$ — ①

$= C(A)$ — ①

5. b) $\phi(i) = 0$ — (1)

$\phi(\infty) = -1$ — (1)

$\phi(1+i) = -1/5 + i\,2/5$ — (1)

Since $\left|-\frac{1}{2} - \left(-\frac{1}{5} + i\frac{2}{5}\right)\right| = \frac{1}{2}$ — (2)

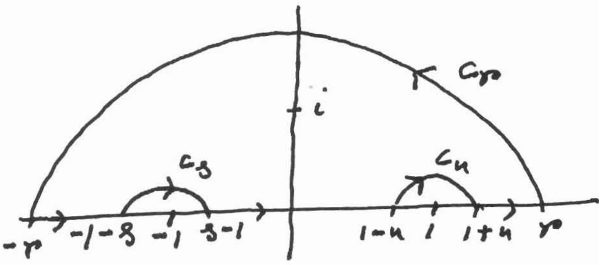
the image is $\{z \,/\, |z + 1/2| = 1/2\}$ — (1)

a) $f(z) = \dfrac{1}{(z+i)(z-i)(z+1)(z-1)}$

$\therefore \operatorname{Res}_{z=1} f(z) = 1/4$ — (1)

$\operatorname{Res}_{z=-1} f(z) = -1/4$ — (1)

$\operatorname{Res}_{z=i} f(z) = \dfrac{1}{2i(-2)} = i/4$ — (1)



$\lim\limits_{\delta \to 0} \displaystyle\int_{C_\delta} f(z)\,dz = -\pi i \operatorname{Res}_{z=-1} f(z) = i\pi/4$ — (1)

$\lim\limits_{u \to 0} \displaystyle\int_{C_u} f(z)\,dz = -i\pi \operatorname{Res}_{z=1} f(z)\,dz = -i\pi/4$ — (1)

$\left|\displaystyle\int_{C_r} f(z)\,dz\right| = \left|\displaystyle\int_0^\pi \frac{i r e^{i\theta}}{r^4 e^{4i\theta} - 1}\,d\theta\right| \le \dfrac{c}{r^3} \to 0$ as $r \to \infty$ — (1)

$\therefore \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \operatorname{Res}_{z=i} f(z) = 2\pi i \times \frac{i}{4} = -\pi/2.$ — (1)

6. a) The fn. is analytic iff $p(z) = (n+1)a_n + \sum_{k=1}^{n} a_k z^k$ does not have any root in $\{z/|z|<1\}$. — (1)

If p has a root z_0 in $\{z/|z|<1\}$ then

$(n+1)a_n = -\sum_{k=1}^{n} a_k z_0^k$ — (1)

$\Rightarrow (n+1)a_n \le \sum_{k=1}^{n} a_k \le n a_n$

$\Rightarrow a_n \le 0$ – contradiction. — (2)

b) In a nbd. of $z_0 = 1/2$ we have

$f(z) = \dfrac{h(z)}{(z-z_0)^2}$ with h analytic around z_0. — (1)

$\therefore f'(z) = \dfrac{h'(z)}{(z-z_0)^2} - \dfrac{2h(z)}{(z-z_0)^3}$ — (1)

$\Rightarrow \dfrac{f'(z)}{f(z)} = -\dfrac{2}{z-z_0} + \dfrac{h'(z)}{h(z)}$ — (1)

$\therefore \dfrac{f'}{f}$ has a simple pole at z_0 with residue -2. — (1)

(1)

Since f has a zero at $z_1 = 0$
then in a nbd. of z_1, $f(z) = z^2 h_1(z)$ s.t. h_1 is analytic & $h_1(0) \neq 0$. — ①
$\therefore f'(z) = 2z h_1(z) + z^2 h_1'(z)$

$\Rightarrow \dfrac{f'(z)}{f(z)} = \dfrac{2}{z} + \dfrac{h_1'(z)}{h(z)}$ — ①

$\therefore \dfrac{f'}{f}$ has a simple pole at $z_1 = 0$ with residue 2. — ①

$\therefore$ By Cauchy's residue theorem $\int_\gamma \dfrac{f'(z)}{f(z)} dz = 0$ — ①

7. a) Since f is cont. $|f(z)| \leq 1$ on $\{z / |z| = 1\}$. — ②
$g(z) = \dfrac{f(z)}{z^n}$ is analytic on $\{z / |z| < 1\}$ as $f(z) = z^n h(z)$ with h analytic. — ①

$\Rightarrow |g(z)| = \dfrac{|f(z)|}{|z|^n} \leq 1 \;\forall z$ with $|z| = 1$ — ①

$\therefore$ By maximum modulus principle $|g(z)| \leq 1$ on $\{z / |z| \leq 1\}$ — ②
$\therefore \; |f(z)| \leq |z|^n \;\forall z, \; |z| < 1$.

b) If $x \geq 0$ then $x = [x] + r_x$, $0 \leq r_x < 1$.
If $x \leq 0$ then $x = -[-x] + s_x$, $-1 < s_x \leq 0$. } — ①
Let $z = x + iy$ with $x \geq 0$, $y \geq 0$ then
$|f(z)| = |f([x] + r_x + i([y] + r_y))|$
$= |f(r_x + i r_y)|$
$\leq M = \sup\{|f(x_0 + iy_0)| / x_0 \in [-1, 1], y_0 \in [-1, 1]\}$ — ③
Same is true if $x \leq 0$ or $y \leq 0$.
So $|f(z)| \leq M \;\forall z \in \mathbb{C}$
$\Rightarrow f$ is constant by Liouville's thm. — ①

c) No. — ① (there does not exist any nonzero entire fn. like that)
Suppose $\exists$ an entire fn. f s.t. $|(z-i)f(z)| \leq M \;\forall z \in \mathbb{C}$.
Then by Liouville's thm. $\exists$ c s.t. $f(z) = \dfrac{c}{z-i}$, $z \neq i$ — ①,
But then f has a simple pole at $z_0 = i$ and hence f is not entire - contradiction. — ①

Note: If somebody does not write $z \neq i$ pl. award 1 mark.

Alt: Yes. — ①
$f(z) = 0 \;\forall z \in \mathbb{C}$. — ②