**Lecture II**

**Low-rank and column rank:** The \( \dim(\text{Row space}) \) and \( \dim(\text{Column space}) \) are called row rank and column rank, respectively.

**Theorem:** The row rank and column rank of any \( m \times n \) matrix are equal.

**Proof:** Let \( A = (a_{ij})_{m \times n} \). Suppose row rank = \( k \) \( \Rightarrow \{v_1, v_2, \ldots, v_k\} \) is a basis for the row space. Let \( \mathbf{v}_r = (b_{r1}, b_{r2}, \ldots, b_{rn}) \), for \( r = 1, 2, \ldots, k \). Since \( \{v_1, \ldots, v_k\} \) is a basis for the row space, each \( \mathbf{v}_r = (a_{r1}, a_{r2}, \ldots, a_{rn}) = \sum_{i=1}^{k} a_{ri} v_i = \sum_{i=1}^{k} a_{ri} (b_{r1}, b_{r2}, \ldots, b_{rn}) \) for some \( a_{ri} \).

\[
\begin{align*}
(3, \ldots, 3) \quad & (3, \ldots, 3) \\
(3,1,1) \quad & (3,1,1)
\end{align*}
\]

To compare \( k \) with the column rank, let us look at the columns of \( A \):

- \( a_{ij} = \sum_{r=1}^{k} \alpha_{ijr} b_{rj} = \alpha_{i1} b_{1j} + \alpha_{i2} b_{2j} + \ldots + \alpha_{ik} b_{kj} \)
- \( a_{2j} = \sum_{r=1}^{k} \alpha_{2jr} b_{rj} = \alpha_{21} b_{1j} + \alpha_{22} b_{2j} + \ldots + \alpha_{2k} b_{kj} \)
- \( a_{mj} = \sum_{r=1}^{k} \alpha_{mrj} b_{rj} = \alpha_{m1} b_{1j} + \alpha_{m2} b_{2j} + \ldots + \alpha_{mk} b_{kj} \)

This implies that

\[
\begin{pmatrix}
\alpha_{ij} \\
\alpha_{2j} \\
\vdots \\
\alpha_{mj}
\end{pmatrix} =
\begin{pmatrix}
\alpha_{i1} \\
\alpha_{i2} \\
\vdots \\
\alpha_{ik}
\end{pmatrix} b_{1j} + \begin{pmatrix}
\alpha_{12} \\
\alpha_{22} \\
\vdots \\
\alpha_{k2}
\end{pmatrix} b_{2j} + \ldots + \begin{pmatrix}
\alpha_{i1} \\
\alpha_{i2} \\
\vdots \\
\alpha_{ik}
\end{pmatrix} b_{kj}
\]

\( C_j \) = jth column of \( A \)

\( C_j \) is a linear combination of \( k \) vectors.

\( \Rightarrow \) Column rank \( \leq \) row rank.

A similar argument will give the converse inequality.

**Definition:** For an \( m \times n \) matrix \( A \), the rank of \( A \) is defined to be the row rank or column rank of \( A \) and it is denoted by \( \text{rank } A \).
Example: Determine the rank of \( A \), where
\[
A = \begin{bmatrix}
1 & 2 & 1 \\
2 & 3 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Solution: By row operations, we get:
\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Therefore, the row rank = column rank = rank \( A = 2 \).

Note that one can also do a sequence of column operations and find the rank.

Solubility of a system (Application):

In one of the previous lectures, we considered the solubility of a system of \( n \) equations with \( n \) unknowns. We will now consider a system of \( m \) linear equations with \( n \) unknowns.

Let \( A \) be an \( m \times n \) matrix. Consider the system \( Ax = b \).

If we consider \( A \) as a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) via \( A: \mathbb{R}^n \to \mathbb{R}^m \), as a consequence of Rank-Nullity Theorem, we get the following results:

1. Existence:

\( Ax = b \) has a solution \( \iff \) \( b \in \text{Range}(A) \)

\( \iff \) \( b \in \text{Span}\{\text{column vectors of } A\} \)

\( \iff \) \( b \in \text{Column space of } A \)

\( \iff \) \( A \) & \((A, b)\) have the same rank

\( \iff \) \( A \) & \((A, b)\) have the same rank

2. Uniqueness:

Suppose \( Ax = b \) has a solution. Then

the solution is unique \( \iff \) \( Ax = 0 \) has only the trivial solution \( x = 0 \).

\( \iff \) \( \ker(A) = N(A) = 0 \)

\( \iff \) \( \text{Nullity}(A) = 0 \)

\( \iff \) \( \text{rank}(A) = n \)

(Note that here \( n \) could be less than \( m \).)

Theorem: Consider the system of equations \( Ax = b \). Let the
rank of \( A \) be \( r \) & \( A \) be an \( m \times n \) matrix (so \( A: \mathbb{R}^n \to \mathbb{R}^m \)).

Then...
(1) If \( Ax = b \) has a solution \( \iff \) \((A, b)\) has rank \( r \).
(2) If \( r = m \) then \( Ax = b \) has a solution for every \( b \in \mathbb{R}^m \).
(3) If \( r = m \), then \( Ax = b \) has a unique solution for every \( b \in \mathbb{R}^m \).
(4) If \( r = m < n \), then for every \( b \in \mathbb{R}^m \), \( Ax = b \) has infinite no. of solutions.
(5) If (i) \( r = m \) (ii) \( r < m \) and (iii) \( r < m \) and \( \forall A \neq 0 \), \( Ax = b \) has a
solution, then it has an infinite number of solutions.
(6) If \( r = n < m \) and \( \forall A \neq 0 \), \( Ax = b \) has a solution, then \( \forall b \), solution is unique.

**Proof:** We have already seen the proofs of (1), (2), (3) and (6) above.
Let us see the proofs of (4) & (5).

**Proof of (4):** First note that, since \( r = m \), the range of \( A \) is \( \mathbb{R}^m \).
Therefore, for every \( b \in \mathbb{R}^m \), there is a solution for \( Ax = b \).
By Rank-Nullity Theorem, \( \dim(N(A)) = n - r > 0 \). Therefore \( N(A) \) has infinite number of solutions and hence, \( Ax = b \)
has infinite number of solutions.

**Proof of (5):** The proof is similar to the proof of (4). \( \forall A \neq 0 \), \( b \in \mathbb{R}^m \)
for some \( x_0 \), then every \( x + x_0 \), \( x \in N(A) \) is a solution.

**Example:** Consider the system
\[
\begin{align*}
2x_1 + x_3 - x_4 + x_5 &= 2 \\
x_1 + x_3 - x_4 + x_5 &= 1 \\
12x_1 + 2x_2 + 6x_3 + 2x_5 &= 12.
\end{align*}
\]

The augmented matrix \((A, b)\) is
\[
\begin{bmatrix}
2 & 0 & 1 & -1 & 1 & 2 \\
1 & 0 & 1 & -1 & 1 & 1 \\
12 & 2 & 8 & 0 & 2 & 12
\end{bmatrix}
\]
After three operations
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 4 & -3 \\
0 & 0 & 1 & -1 & 10
\end{bmatrix}
\]
\(\Rightarrow\) \( \text{Rank}(A) = \text{Rank}(A, b) = 3 \). (Here \( A : \mathbb{R}^5 \to \mathbb{R}^3 \))

Therefore, by Rank-Nullity Theorem,
\[
\dim(N(A)) = \dim(\text{dim}(A) - \text{dim}(\text{range}) = 5 - 3 = 2
\]
So the set of solutions of the system is the translation of \( N(A) \).
Let us find the set of solutions. Note that
\[
\begin{align*}
  x_1 &= 1 \\
  x_2 + 4x_4 - 3x_5 &= 0 \\
  x_3 - x_4 + x_5 &= 0
\end{align*}
\]
\[
\Rightarrow \quad \begin{align*}
  x_2 &= -4x_4 + 3x_5 \\
  x_3 &= x_4 - x_5
\end{align*}
\]
If we choose \( x_4 \) and \( x_5 \) arbitrarily, the set of solutions can be written as
\[
\{ (1, -4x_4 + 3x_5, x_4 - x_5, x_4, x_5) : x_4, x_5 \in \mathbb{R} \}
\]
\[
= \{ (1, 0, 0, 0, 0) + x_4 (0, -4, 1, 1, 0) + x_5 (0, 3, -1, 0, 1) : x_4, x_5 \in \mathbb{R} \}
\]
\[
= (1, 0, 0, 0, 0) + \text{span} \{(0, -4, 1, 1, 0), (0, 3, -1, 0, 1)\}
\]
\[
= (1, 0, 0, 0, 0) + N(A).
\]

Example 2: Consider the system:
\[
\begin{align*}
  x_1 + 2x_2 + 4x_3 + x_4 &= 4 \\
  2x_1 - x_3 - 3x_4 &= 4 \\
  x_1 - 2x_2 - x_3 &= 0 \\
  3x_1 + x_2 - x_3 - 5x_4 &= 5
\end{align*}
\]
The augmented matrix is,
\[
\begin{bmatrix}
  1 & 2 & 4 & 1 & 4 \\
  2 & 0 & -1 & -3 & 4 \\
  1 & -2 & -1 & 0 & 0 \\
  3 & 1 & -1 & 0 & 5
\end{bmatrix}
\]
After few iterations,
\[
\begin{bmatrix}
  1 & 0 & 0 & -1 & 2 \\
  0 & 0 & 1 & 1 & 0 \\
  0 & 0 & 0 & 0 & 8 \\
  0 & 0 & 0 & 0 & 7
\end{bmatrix}
\]
Here \( \text{rank}(A) = 3 < \text{rank}(A, b) = 4 \). So the system cannot have a solution. In other words, we say that the system is inconsistent.