In the previous lecture, we have seen that if an $n \times n$ matrix has $n$ distinct eigenvalues then the matrix is diagonalizable. In this lecture we will see that any (real) symmetric matrix is diagonalizable.

We will first see that a square matrix whose row vectors or column vectors are orthonormal has some interesting properties.

The proof of the following lemma will be discussed in the Tutorial class.

Lemma: Let $A$ be an $n \times n$ matrix. Then the following statements are equivalent.

1. The column vectors are orthonormal.
2. $A^T A = I_n$
3. $A^T = A^{-1}$
4. $A A^T = I_n$
5. $\|Ax\| = \|x\| \, \forall x \in \mathbb{R}^n$
6. $\langle Ax, Ay \rangle = \langle x, y \rangle, \, \forall x, y \in \mathbb{R}^n$
7. The row vectors are orthonormal.

Definition: A square matrix $A$ is called an orthogonal matrix if a matrix satisfies one of the statements of the above lemma; in particular, $AA^T = A^T A = I$.

Examples: The matrices $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \begin{bmatrix} \cos \theta & 0 \\ 0 & \cos \theta \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are orthogonal.

So far we have been dealing with matrices whose entries are real.
If A is a complex matrix, then we can define eigen-values and eigen-vectors and do diagonalization as we did for the real case. In the complex case, the eigen-values are in $\mathbb{C}$ & the eigen-vectors are in $\mathbb{C}^n$. In $\mathbb{C}^n$, we define $ \langle \cdot, \cdot \rangle$ as follows: $\langle u, v \rangle = u^* \overline{v}$

**Theorem:** Let A be an $n \times n$ (real) matrix. If A is symmetric, then A has n real eigen-values.

**Proof:** The characteristic polynomial $|A - \lambda I|$ has n roots in $\mathbb{C}$, each root is an eigen-value of A.

Let $\lambda \in \mathbb{C}$ be any eigen-value & $u \in \mathbb{C}^n$ be a corresponding eigenvector of A. Then $Au = \lambda u$ (i.e. $u^* A = \lambda u^*$ (as $A \in \mathbb{R}^{n \times n}$))

$\Rightarrow \overline{u}^* A = \overline{u}^* \lambda u$ (by taking complex conjugation both sides)

$\Rightarrow \overline{u}^* Au = \overline{u}^* \overline{\lambda} u$, but we know $\overline{u}^* Au = \lambda u^* u$

$\Rightarrow \overline{\lambda} u^* u = \lambda u^* u \Rightarrow \lambda (\|u\|^2) = \lambda (\|u\|^2) \Rightarrow \lambda = \overline{\lambda}$

**Remark:** If A is a real symmetric matrix then its eigen-values are in $\mathbb{R}$. This can be proved as follows. Let $\lambda$ be an e.v. Since it is real & $(A - \lambda I)$ is real and non-invertible, $\exists u \in \mathbb{R}^n, s.t. (A - \lambda I)u = 0$.

**Theorem:** Let A be a real symmetric matrix. Then if an orthogonal matrix $Q$ s.t. $A = QDQ^{-1}$ where D is a diagonal matrix and the diagonal entries of D are the eigen-values of A.

**Proof:** Let A be $n \times n$ real symmetric matrix. We prove the theorem by induction on n. For n = 1, the result is obvious.

Let m assume that the result is true for all $(n-1) \times (n-1)$ matrices. Let A be an $n \times n$ matrix. By the previous theorem, A has a real eigen-value; call it $\lambda_1$. Let $A\lambda_1 = \lambda_1$, & $\|\lambda_1\| = 1$.

Let $\{X, V_2, \ldots, V_n\}$ be an orthonormal basis of $\mathbb{R}^n$. This can be obtained by G-S process.
Define \( \Phi_1 = [X_1, V_2, \ldots, V_n] \). Note that \( \Phi_1 \) is orthogonal. Moreover, \( \Phi_1 A \Phi_1 \) is a real symmetric matrix because,

\[
(\Phi_1^{-1} A \Phi_1)^T = (\Phi_1^T A \Phi_1)^T = (\Phi_1^T A \Phi_1) = \Phi_1^{-1} A \Phi_1.
\]

Let us evaluate first column of \( \Phi_1^{-1} A \Phi_1 \). By symmetry we will know the first row.

The first column is given by

\[
(\Phi_1^{-1} A \Phi_1) (e_1) = (\Phi_1^{-1} A) \Phi_1 e_1 = (\Phi_1^{-1} A) x_1 = \Phi_1^{-1} (\lambda_1 x_1) = \lambda_1 \Phi_1 x_1 = \lambda_1 e_1
\]

because \( \Phi_1 e_1 = x_1 \).

Therefore,

\[
\Phi_1^{-1} A \Phi_1 = \begin{bmatrix}
\lambda_1 & 0 \\
0 & A_1
\end{bmatrix}
\]

where \( A_1 \) is an \( (n-1) \times (n-1) \) symmetric square matrix.

By induction, if an orthogonal matrix \( \Phi_2 \) s.t. \( \Phi_2^{-1} A \Phi_2 = D_1 \), an \( (n-1) \times (n-1) \) diagonal matrix.

Claim: if \( \Phi \) such that \( \Phi^{-1} A \Phi = \begin{bmatrix} \lambda_1 & 0 \\ 0 & D \end{bmatrix} \).

Let us find \( \Phi \). Note that

\[
\begin{bmatrix}
\lambda_1 & 0 \\
0 & D_1
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & D_1
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \Phi_2^T \end{bmatrix} \begin{bmatrix}
\lambda_1 & 0 \\
0 & D_1
\end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Phi_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & \Phi_2 \end{bmatrix} (\Phi_1^{-1} A \Phi_1) \begin{bmatrix} 1 & 0 \\ 0 & \Phi_2 \end{bmatrix}
\]

\[
= \left( \Phi_1 \begin{bmatrix} 1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \right)^T \begin{bmatrix} 1 & 0 \\ 0 & \Phi_2 \end{bmatrix}
\]

one can easily verify that the matrix \( \Phi = \Phi_1 \begin{bmatrix} 1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \) is an orthogonal matrix. \( \square \)