Lecture 3

Determinants: There are different ways to define determinants. Each one has its advantages. We will define using the concept permutation.

Permutation: Let $S$ be any finite set. Here we take $S = \{1, 2, \ldots, n\}$. Any one-one onto mapping of $S$ to itself is called permutation.

If $\sigma$ is a permutation then $\sigma^{-1}$ is also a permutation. Also, given a permutation $\sigma$ and $\tau$, the composition $\sigma \circ \tau$ is also a permutation.

We denote the set of all permutations on $\{1, 2, \ldots, n\}$ by $S_n$.

Examples: Let $\sigma, \phi \in S_4$ be defined by $\sigma = (1 \ 2 \ 3 \ 4) \ 2$ $\phi = (1 \ 2 \ 3 \ 4) \ 3$. Then $\sigma \circ \phi = (1 \ 2 \ 3 \ 4) \ 4$ and $\phi \circ \sigma = (1 \ 2 \ 3 \ 4) \ 2$ $\sigma^{-1} = (1 \ 2 \ 3 \ 4) \ 4$, clearly $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = (1 \ 2 \ 3 \ 4)$.

Transposition: A permutation is called transposition if it moves exactly two points in $S$.

We need the following two results which we state without proof:

Theorem: 1. Every permutation can be written as a product (composition) of transpositions.

2. If $\sigma$ is a permutation and $\sigma_1, \sigma_2, \ldots, \sigma_r$ are transpositions such that $\sigma = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_r$, then $r$ and $\sigma$ are both even or odd. (Note that here $\sigma_1, \ldots, \sigma_r$ need not be distinct)

Ex: Let $\sigma = (1 \ 2 \ 3 \ 4)$. Then $\sigma = (1 \ 2 \ 3 \ 4) \circ (2 \ 4 \ 1 \ 3) \circ (1 \ 2 \ 3 \ 4) \circ (2 \ 4 \ 1 \ 3)$.

We write $\sigma = (13)(4)(2)$, note that $\sigma = (21)(23)(24)$.

Even or odd permutation: A permutation $\sigma$ is called an even permutation if it can be written as a product of an even number of transpositions. Otherwise, it is called an odd permutation.
The identity permutation $I$ is even and every transposition is odd.

**Definition** Define $\text{Sign}(\sigma) = 1$ if $\sigma$ is even & $\text{Sign}(\sigma) = -1$ if $\sigma$ is odd.

**Determinants:** Let $A = (a_{ij})$ be an $n \times n$ matrix. Its determinant, denoted by $|A|$, is:

$$\det A = |A| = \sum_{\sigma \in S_n} \text{Sign}(\sigma) a_{1 \sigma(1)} a_{2 \sigma(2)} \cdots a_{n \sigma(n)}$$

(Note that $S_n$ has $n!$ elements).

**Example:** Let $A = (a_{11}, a_{12})$. (Here $S_2 = \{ I, (1, 2) \}$, so $|A| = (\text{sign } I) a_{11} a_{22} + \text{sign } (1, 2) a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21}$.)

This definition is not very convenient for computing determinants. However, several properties of determinants can be derived easily from this definition.

**Properties of the determinants:** Let $A = (a_{ij})$ & $B = (b_{ij})$ be $n \times n$ matrices. Then $\det AB$ is obtained by interchanging two rows of $A$ then $|A| = -|B|$.  

**Proof:** Suppose $b_{pq} = a_{pq}$, $b_{eq} = a_{eq}$ & $a_{ij} = b_{ij}$, $1 + p, 2 + q$.

Consider $1$-transposition $\tau = (p q)$. Note that $S_n = \{ \sigma \tau : \sigma \in S_n \}$. Hence:

$$|B| = \sum_{\sigma \in S_n} \text{Sign}(\sigma) b_{1 \sigma(1)} b_{2 \sigma(2)} \cdots b_{n \sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{Sign}(\sigma) \text{Sign}(I) b_{1 \sigma(1)} b_{2 \sigma(2)} \cdots b_{n \sigma(n)}$$

$$= (\text{Sign } I) \sum_{\sigma \in S_n} \text{Sign}(\sigma) a_{1 \sigma(1)} a_{2 \sigma(2)} \cdots a_{n \sigma(n)}$$

$$= -|A|. \quad \square$$
Cor: (P2) If A has two identical rows then \( |A| = 0 \).

Proof: Let B be the matrix obtained by interchanging those two identical rows. Then \( A = B \). Since \( |A| = -|B| \) by (P1), \( |A| = 0 \).

The proofs of the following properties (P3) and (P4) are similar to the proof of (P1).

Prop: (P3) If \( B \) is obtained by multiplying a row of \( A \) by a constant \( c \), then \( |B| = c |A| \).

Prop: (P4) Suppose \( C = (c_{ij}) \). Further assume that \( A, B \) and \( C \) differ only in the \( k \) th row for some \( k \) s.t. \( c_{kj} = a_{kj} + b_{kj} \).

Then \( |C| = |A| + |B| \).

Prop: (P5) If \( B \) is obtained by adding \( c \) times the \( p \) th row of \( A \) to its \( q \) th row then \( |A| = |B| \).

Proof: Note that \( b_{ij} = a_{ij} + c a_{pj} \) if \( i \) and \( b_{ij} = a_{ij} + c a_{pj} \) if \( i \neq q \). Let \( C = (c_{ij}) \) where, for all \( j \), \( c_{ij} = (i\neq p) a_{ij} + c a_{pj} \).

Then \( A, B \) and \( C \) differ only in the \( q \) th row and \( b_{ij} = a_{ij} + c a_{ij} \).

Hence by (P4), \( |B| = |A| + |C| \). Further \( |C| = 0 \) by (P3) & (P).