As mentioned earlier, evaluating a determinant from its definition is not easy. We will derive two more properties of the determinant which will provide an inductive method for computing 1×1 determinants.

**Lemma:** Let \( A = \begin{pmatrix} B & a_{1n} \\ a_{21} & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \ddots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \) where \( B = (a_{ij})_{n-1 \times n-1} \). Then \( |A| = |B| \).

**Proof:** By definition \( |A| = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \).

\[ = \sum_{\sigma \in S_n, \sigma(n)=n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n-1,\sigma(n-1)} \] (\( \therefore \sigma(n) \neq n \) then \( a_{n\sigma(n)} = 0 \))

\[ = \sum_{\sigma \in S_n, \sigma(n)=n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n-1,\sigma(n-1)} \] (\( \therefore \sigma(n) = n \))

\[ = \sum_{\sigma \in S_n, \sigma(n)=n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n-1,\sigma(n-1)} \] (\( \therefore S_{n-1} = \{ \sigma \in S_n : \sigma(n) = n \} \)).

**Theorem:** Let \( A = (a_{ij}) \) and let \( A_{ij} \) denote the matrix obtained by removing the \( i \)th row and the \( j \)th column of \( A \). Then

\[ |A| = \sum_{j=1}^{n} (-1)^{i+j} |A_{ij}|. \]

**Proof:** Fix an \( i \in \{1,2,\ldots,n\} \). Define

\[ A_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad A_3 = \cdots \]

Note that \( A_{ij} = (A)_{ij} = (A_1)_{ij} + (A_2)_{ij} + \cdots + (A_n)_{ij} \) and other rows of matrices \( A, A_1, A_2, \ldots, A_n \) are the same. Hence by (PA), \( |A| = \sum_{j=1}^{n} |A_{ij}| \).

**Claim:** \( |A_{ij}| = (-1)^{i+j} |A_{ij}| \), where \( A_{ij} \) is as defined in the statement of the result.
Note that

\[
A_j = \begin{pmatrix}
    \ldots & a_{ij} & \ldots \\
    \ldots & \ddots & \ldots \\
    \ldots & \ldots & a_{nn}
\end{pmatrix}
\]

\[\rightarrow\]

\[
\begin{pmatrix}
    A_{ij} & \ldots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \ldots & A_{nn}
\end{pmatrix}
\]

\[\rightarrow\]

\[
\begin{vmatrix}
    A_{ij} & \ldots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \ldots & A_{nn}
\end{vmatrix} = (\pm 1) \begin{vmatrix}
    a_{ij} & \ldots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \ldots & a_{nn}
\end{vmatrix}
\]

Determinant method of finding inverse:

Take some \( k \neq i \) and obtain the matrix \( B \) by replacing the
\( i \)th row of \( A \) with the \( k \)th row of \( A \) (by keeping the \( k \)th row
as it is). Then we get,

\[
\sum_{j=1}^{n} (\pm 1)^{i+j} A_{kj} |A_{ij}| = |B| = 0
\]

(As two rows of \( B \) are same).

So, \( \sum_{j=1}^{n} A_{kj} C_{ij} = |A| \) and \( \sum_{j=1}^{n} A_{kj} C_{ij} = 0 \).

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
    C_{i1} \\
    C_{i2} \\
    \vdots \\
    C_{in}
\end{pmatrix}
\]

\[
= |A| \begin{pmatrix}
    0 \\
    0 \\
    \vdots \\
    0
\end{pmatrix}
\]

\[
\begin{pmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    a_{21} & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
    C_{i1} & C_{i2} & \ldots & C_{in} \\
    C_{i2} & C_{i2} & \ldots & C_{i2} \\
    \vdots & \vdots & \ddots & \vdots \\
    C_{in} & C_{in} & \ldots & C_{in}
\end{pmatrix}
\]

= |A| I_n

\[\Rightarrow\]

\[A^{-1} = \begin{pmatrix}
    C_{i1} & C_{i2} & \ldots & C_{in} \\
    C_{i2} & C_{i2} & \ldots & C_{i2} \\
    \vdots & \vdots & \ddots & \vdots \\
    C_{in} & C_{in} & \ldots & C_{in}
\end{pmatrix}\]

\[\Rightarrow\]

\[A (C_{ij}) = |A|, \text{ the matrix } (C_{ij}) \text{ is called the matrix of cofactor of } A\]
The matrix \((C_{ij})^T\) is called the (classical) adjoint (or adjugate) of \(A\) and it is denoted by \(\text{Adj} \ A\). Thus

\[ A \ (\text{Adj} \ A) = |A| \ I_n. \]

Therefore the following result is immediate from Kii's fact.

**Theorem:** Let \(A\) be an \(n \times n\) matrix. If \(A\) is invertible, then

\[ A^{-1} = \frac{1}{|A|} (\text{Adj} \ A). \]

**Cramer's Rule for Solving System of Linear Equations:**

The following result is about solvability of a system of linear equations.

**Theorem:** Let \(A\) be an \(n \times n\) matrix. Then the following statements are equivalent:

1. \(|A| \neq 0\)
2. \(A\) is invertible
3. \(Ax = b\) has a unique solution for every \((n \times 1)\) matrix \(b\)
4. \(Ax = b\) has a solution for every \(b\)

**Proof:** (1) \(\Rightarrow\) (2): We have already seen the proofs of these implications.

(2) \(\Rightarrow\) (3): For \(b\), choose \(x = A^{-1}b\) and note that \(Ax = A(A^{-1}b) = b\) and \(A^{-1}b\) has to be the only solution for \(Ax = b\).

(3) \(\Rightarrow\) (4): Obvious.

(4) \(\Rightarrow\) (2): For \(b_i = \begin{pmatrix} 0 \\ \delta_i \\ b \end{pmatrix}\) at \(i\), \(U_i\) is the \(i\)-th component of \(b\).

Hence \(AB = I_n\), where \(B = (U_1, U_2, \ldots, U_n)\).

**Corollary:** Let \(A\) be an \(n \times n\) matrix. Then the following are equivalent:

1. \(A\) is invertible
2. \(Ax = 0\) has only the trivial solution \(x = 0\).
Proof: (1) \( \Rightarrow \) (2) follows from the previous result.

(2) \( \Rightarrow \) (3) of the previous result: Suppose \( A u_1 = A u_2 = b \) for some \( b \) and \( u_1 \neq u_2 \). Then \( A(u_1 - u_2) = 0 \) where \( u_1 - u_2 \neq 0 \) which is a contradiction.

**Cramer's Rule:** Let \( A x = b \) be a system of \( n \) linear equations in \( n \) unknowns \( x_1, x_2, \ldots, x_n \). Then the system has a unique solution given by

\[
x_j = \frac{|C_j|}{|A|}, \quad j = 1, 2, \ldots, n.
\]

where \( C_j \) is the matrix obtained from \( A \) by replacing the \( j \)th column with the column matrix \( b = (b_1, b_2, \ldots, b_n)^t \).

Proof: \( \Rightarrow \) \( |A| \neq 0 \), then \( A \) is invertible and \( x = A^{-1}b \) is the unique solution of \( A x = b \),

i.e., \( x = \frac{1}{|A|} (\text{adj } A) b \)

\[
\Rightarrow x_j = \frac{1}{|A|} \ b_1 C_{1j} + b_2 C_{2j} + \ldots + b_n C_{nj}
\]

\[
= \frac{|C_j|}{|A|}.
\]

\( \Box \)