We have seen that if \( |A| = 0 \) and the system \( Ax = 0 \) has more than one solution then it has infinite number of solutions. We will see that, in this case, the set of solutions has a structure, called a vector space. We will first introduce the notion of a vector space which is a generalization of the algebraic structure present in \( \mathbb{R}^3 \). The study of matrices will be elaborated within this framework.

**Definition:** A real vector space is a nonempty set \( V \) with two algebraic operations that satisfy the following rules:

(A) There is an operation called **addition** that associates to every pair of elements \( x, y \in V \) a unique element \( x + y \in V \) such that:

1. \( x + y = y + x \)
2. \( x + (y + z) = (x + y) + z \)
3. If \( x \) is a unique element in \( V \), called 0, such that \( x + 0 = 0 + x = x \)
4. For any \( x \in V \), there is an element \( -x \in V \) such that \( x + (-x) = (-x) + x = 0 \) (where 0 is called additive identity and \( -x \) is called additive inverse of \( x \)).

(B) There is an operation called **scalar multiplication** that associates to each \( x \in V \) and \( \alpha \in \mathbb{R} \) a unique element \( \alpha x \in V \) such that:

5. \( \alpha (x + y) = \alpha x + \alpha y \)
6. \( \alpha (\beta x) = (\alpha \beta) x \)
7. \( (\alpha + \beta) x = \alpha x + \beta y \)
8. \( 1 \cdot x = x \) for all \( x \in V \)

The elements of a vector space are called Vectors.

**Examples:**

1. The set of reals \( \mathbb{R} \) is a Vector space.

2. Let \( V = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\} \). Define \((x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\) and
for \( x \in \mathbb{R}, \quad x(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) \). Then \( V \) is a vector space.

3. Let \( V = M(n, \mathbb{R}) = \{ A \} \) set of all \( n \times n \) matrices with real entries. Define \((a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})\) and \( x(a_{ij}) = (xa_{ij})\). Then \( V \) is a vector space.

4. Let \( A \) be an \( m \times n \) matrix. Then \( V = \{ x : Ax = 0 \} \) is a vector space.

5. Let \( V \) be the set of all \( n \times n \) matrices with real entries and degree less or equal to \( n-1 \). For \( P(x) = \sum_{i=0}^{n-1} a_i x^i \) and \( Q(x) = \sum_{i=0}^{n-1} b_i x^i \), \( x \in \mathbb{R} \), define \((P + Q)(x) = \sum_{i=0}^{n-1} (a_i + b_i) x^i \) and \((xP)(x) = \sum_{i=0}^{n-1} a_i x^i \). Then \( V \) is a vector space with additive identity as the zero polynomial. Note that a sum of two \( n \times n \) degree polynomials may not be a \( n \) degree polynomial, the set of all \( n \) degree polynomials is not a vector space.

**Definition:** A complex vector space is where we can choose \( \mathbb{C} \) and the scalar multiplication satisfies (V) – (viii).

For example \( \mathbb{C}^n \), \( M(n, \mathbb{C}) \) etc. are complex vector spaces.

From now onwards by \( V \) we will mean a real vector space.

**Theorem:** Let \( V \) be a vector space and \( \mathbb{R} \leq V \). Then

1. 0 \( \cdot x = 0 \)
2. Additive identity and additive inverse are unique
3. \(-1 \cdot x = -x\)
4. \( x \cdot 0 = 0 \) for \( x \in \mathbb{R} \)
5. If \( x \cdot x = 0 \) then either \( x = 0 \) or \( x = 0 \).

**Proof:** we will prove only 1). Others are similar. We have

\[
0x = -(0x) + 0x = -(0x) + (0 + 0)x = -(0x) + 0 + 0x = 0x
\]
Definition: Let \( W \) be a nonempty subset of a vector space \( V \). Then \( W \) is called a vector subspace (or simply subspace) of \( V \) if \( W \) is a vector space under the operations defined in \( V \).

In order to show a subset a subspace, there is no need to verify the rules (i)-(viii) of the vector space. It is enough to check that (i) \( 0 \in W \) (ii) \( w_1 + w_2 \in W \), \( w_1, w_2 \in W \) (iii) \( \alpha w \in W \) for \( \alpha \in \mathbb{R} \) \& \( w \in W \), the other rules are satisfied automatically.

Examples:
1. The set \( A = \{ (x, y, z) : x + y + z = 1 \} \) in \( \mathbb{R}^3 \) is a subspace.
2. The set of all polynomials of degree \( \leq n - 2 \) is a subspace of all polynomials of degree \( \leq n - 1 \).
3. \( \{ 0 \} \) is always a subspace of any vector space.
4. Set of all polynomials with nonnegative coefficients of degree \( \leq n - 1 \) is not a subspace of all polynomials of degree \( \leq n - 1 \).
5. The set of points on a straight line which is not passing through origin is not a subspace of \( \mathbb{R}^2 \).
6. The set of points on a circle is not a subspace of \( \mathbb{R}^2 \).
7. The union of two straight lines passing through origin is not a subspace (unless both are same).