**Linear span:**

1. Take \((1,1) \in \mathbb{R}^2\) and consider \(\{\lambda (1,1) : \lambda \in \mathbb{R}\}\). This is a straight line passing through origin and hence it is a subspace of \(\mathbb{R}^2\).

2. Note that \(\{\lambda (1,1) + \beta (1,0) : \lambda, \beta \in \mathbb{R}\} = \{\lambda (1,0) + \beta (0,1) : \lambda, \beta \in \mathbb{R}\} = \mathbb{R}^2\).

3. In \(\mathbb{R}^3\), \(\{\lambda (1,1,1) + \beta (2,1,3) : \lambda, \beta \in \mathbb{R}\}\) is a plane passing through origin and this is a subspace of \(\mathbb{R}^3\). (The equation of the plane is \(2x - y = 2\).

In these examples we see that one or two elements generate subspaces. We will define this concept formally below.

**Definition:** Let \(S = \{u_1, u_2, \ldots, u_n\}\) be a subset of a vector space \(V\). The linear span of \(S\) is the set defined by

\[
\text{Span}(S) = \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n : \alpha_i \in \mathbb{R}, 1 \leq i \leq n\}.
\]

If \(S\) is empty, we define \(\text{Span}(S) = \{0\}\). The combination \(\alpha_1 u_1 + \cdots + \alpha_n u_n\) is called a linear combination of \(u_i\)’s.

**Example:** Is \((4, 5, 5)\) a linear combination of \((1,2,3), (-1,1,4), (3,3,2)\)?

To answer, one has to find \(\alpha, \beta, \gamma\) such that \(\alpha (1,2,3) + \beta (-1,1,4) + \gamma (3,3,2) = (4, 5, 5)\). In fact, \(\alpha (1,2,3) + \beta (-1,1,4) + \gamma (3,3,2) = (4, 5, 5)\).

If \(S\) is any arbitrary subset of \(V\), the linear span is defined as follows:

\[
\text{Span}(S) = \{\alpha_1 u_1 + \cdots + \alpha_n u_n : \alpha_i \in \mathbb{R}, u_i \in S\},
\]

the collection of all (finite) linear combinations of elements of \(S\).

**Proposition:** Let \(S\) be any non-empty subset of a vector space \(V\). Then \(\text{Span}(S)\) is a subspace of \(V\). In fact, it is the smallest subspace of \(V\) containing \(S\). (We say that \(\text{Span}(S)\) is spanned by \(S\)).

**Proof:** It is easy to verify that \(\text{Span}(S)\) is a subspace and \(S \subseteq \text{Span}(S)\). If \(W\) is a subspace of \(V\) and \(S \subseteq W\), then every linear combination of elements of \(S\) belongs to \(W\), i.e. \(\text{Span}(S) \subseteq W\).
Linearly dependent:

Note that span \( \{(1,1), (2,2)\} \) is a proper subspace of \( \mathbb{R}^2 \) but span \( \{(1,1), (0,1)\} = \mathbb{R}^2 \). In the first case \( (2,2) = 2(1,1) \), i.e., \( (2,2) \) depends on \( (1,1) \) but in the second case one element doesn't depend on the other.

**Definition:** Let \( V \in \mathbb{V} \) and \( \{v_i, v_2, ..., v_k\} \subseteq V \). We say that \( V \) is linearly dependent \( (L.D.) \) on \( v_i, v_2, ..., v_k \) if there exist \( \alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R} \) s.t. \( \sum_{i=1}^{k} \alpha_i v_i = 0 \).

**Example:** We have seen that \( (4,5,5) = 3(1,2,3) + (-1)(-1,1,4) + 0(3,3,2) \). Note that in this case, \( (-1,1,4) = 3(1,2,3) - (4,5,5) + 0(3,3,2) \). So if \( V \) is L.D. on \( v_1, v_2, ..., v_k \), then \( I \) is such that \( \alpha_i \) in L.D. on \( v_1, v_2, ..., v_i, v_{i+1}, v_{i+2}, ..., v_k, v \). So in a slightly different point of view we say that \( \{v_1, v_2, ..., v_k, v\} \) is a L.D. set if there exist an element which can be written as a linear combination of the rest of the elements.

**Definition:** We say that a set \( \{v_1, v_2, ..., v_n\} \) is L.D. if \( \exists \alpha_i \in \mathbb{R}, i \leq i \leq n, \) not all zero such that \( \sum_{i=1}^{n} \alpha_i v_i = 0 \). If \( \{v_1, v_2, ..., v_n\} \) is not L.D. then it is called linearly independent (L.I).

In order to verify a given set \( \{v_1, v_2, ..., v_n\} \) is L.D. or L.I., we consider the equation

\[
\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0
\]

In case, \( \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0 \) is the only solution then the set is L.I. otherwise it is L.D.

**Example:** Since \( 3(1,2,3) - (-1,1,4) - (4,5,5) = 0 \), \( \{v_1, v_2, ..., v_k\} \) is L.D.

2. Consider the set \( \{(2,0,1), (3,1,0), (5,6,4)\} \) and...
\( \alpha (2,0,0) + \beta (3,1,0) + \gamma (5,6,4) = 0. \)

It is easy to verify that \( \alpha = \beta = \gamma = 0. \) Do the given set is L.I.

3. Verify that \( \{ (1,1,1), (1,1,0), (0,0,1) \} \) is L.I.

**Proposition:** Let \( V \) be a vector space and \( S \subseteq V. \)

1. If \( S \) is L.I then \( 0 \notin S. \)
2. If \( S \) is L.I., then every non-empty subset of \( S \) is L.I.
3. If \( S \) is L.D. then every set containing \( S \) is also L.D.

Consider the sets \( \{ (1,0,0), (0,1,0) \} \) and \( \{ (1,0,0), (0,1,0), (0,0,1) \} \).

These two sets are L.I. However, the set \( \{ (1,0,0), (0,1,0), (0,0,1) \} \)

is something special because, it spans the entire space \( \mathbb{R}^3. \)

**Definition:** A subset \( B = \{ v_1, v_2, \ldots, v_n \} \) is a basis of \( V \) if

(i) \( B \) is L.I.
(ii) \( \text{Span}(B) = V. \)

Recall that the condition (ii) says that every element of \( V \)

can be expressed as a linear combination of elements of \( B. \)

**Examples:**

1. The set \( \{ e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1) \} \) is a basis of \( \mathbb{R}^3. \) The basis \( \{ e_1, e_2, e_3 \} \) is called the standard basis for \( \mathbb{R}^3.

2. The set \( \{ (1,1,0), (0,1,1), (1,0,1) \} \) is L.D. Hence it is not a basis of \( \mathbb{R}^3. \) Note the set can't span \( \mathbb{R}^3.

3. The set \( \{ (1,0,0), (0,1,0), (0,0,1) \} \) spans \( \mathbb{R}^3 \) but it is not a basis because it is not L.I.

4. The set \( \{ (1,1,1), (1,1,0), (1,0,0) \} \) is a basis for \( \mathbb{R}^3. \) This is different from the standard basis of \( \mathbb{R}^3.

5. The set \( \{ 1, x, x^2 \} \) is a basis for \( P_2, \) the space of all polynomials of degree \( \leq 2. \)

**Remark:** If \( \{ v_1, v_2, \ldots, v_n \} \) is a basis for \( V, \) then any \( v \in V \) is a unique linear combination \( v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n. \)