LECTURE 5: COMPLEX LOGARITHM AND TRIGONOMETRIC
FUNCTIONS

Let \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). Recall that \( \exp : \mathbb{C} \to \mathbb{C}^* \) is surjective (onto), that is, given \( w \in \mathbb{C}^* \) with \( w = \rho(\cos \phi + i \sin \phi), \rho = |w|, \phi = \text{Arg} \ w \) we have \( e^z = w \) where \( z = \ln \rho + i \phi \) (ln stands for the real log). Since exponential is not injective (one one) it does not make sense to talk about the inverse of this function. However, we also know that \( \exp : H \to \mathbb{C}^* \) is bijective. So, what is the inverse of this function? Well, that is the logarithm. We start with a general definition

**Definition 1.** For \( z \in \mathbb{C}^* \) we define \( \log z = \ln |z| + i \arg z \).

Here \( \ln |z| \) stands for the real logarithm of \( |z| \). Since \( \arg z = \text{Arg} \ z + 2k\pi, k \in \mathbb{Z} \) it follows that \( \log z \) is not well defined as a function (it is multivalued), which is something we find difficult to handle. It is time for another definition.

**Definition 2.** For \( z \in \mathbb{C}^* \) the principal value of the logarithm is defined as \( \text{Log} \ z = \ln |z| + i \text{Arg} \ z \).

Thus the connection between the two definitions is \( \text{Log} \ z + 2k\pi = \log z \) for some \( k \in \mathbb{Z} \). Also note that \( \text{Log} : \mathbb{C}^* \to H \) is well defined (now it is single valued).

**Remark:** We have the following observations to make,

1. If \( z \neq 0 \) then \( e^{\text{Log} \ z} = e^{\ln |z| + i \text{Arg} \ z} = z \) (What about \( \text{Log} \ (e^z) \)?)
2. Suppose \( x \) is a positive real number then \( \text{Log} \ x = \ln x + i \text{Arg} \ x = \ln x \) (for positive real numbers we do not get anything new).
3. \( \text{Log} \ i = \ln |i| + i \frac{\pi}{2} = \frac{i\pi}{2} \), \( \text{Log} \ (-1) = \ln |-1| + i\pi = i\pi \), \( \text{Log} \ (-i) = \ln |-i| + i \frac{\pi}{2} = -\frac{i\pi}{2} \), \( \text{Log} \ (-e) = 1 + i\pi \) (check!)
4. The function \( \text{Log} \ z \) is not continuous on the negative real axis \( \mathbb{R}^- = \{z = x + iy : x < 0, y = 0\} \). (Unlike real logarithm, it is defined there, but useless). To see this consider the point \( z = -\alpha, \alpha > 0 \). Consider the sequences \( \{a_n = \alpha e^{(\pi - \frac{\pi}{n})}\} \) and \( \{b_n = \alpha e^{(-\pi + \frac{\pi}{n})}\} \). Then \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = z \) but \( \lim_{n \to \infty} \text{Log} \ a_n = \lim_{n \to \infty} \ln \alpha + i(\pi - \frac{\pi}{n}) = \ln \alpha + i\pi \) and \( \lim_{n \to \infty} \text{Log} \ b_n = \ln \alpha - i\pi \).
5. \( z \to \text{Log} \ z \) is analytic on the set \( \mathbb{C}^* \setminus \mathbb{R}^- \). Let \( z = re^{i\theta} \neq 0 \) and \( \theta \in (-\pi, \pi) \). Then \( \text{Log} \ z = \ln r + i\theta = u(r, \theta) + iv(r, \theta) \) with \( u(r, \theta) = \ln r \) and \( v(r, \theta) = \theta \).
Then \( u_r = \frac{1}{r} v_\theta \) and \( v_r = -\frac{1}{r} u_\theta \). Thus the CR equations are satisfied.

Since \( u_r, u_\theta, v_r, v_\theta \) are continuous the result follows from a previous theorem regarding converse of CR equations.

(6) The identity \( \log (z_1 z_2) = \log z_1 + \log z_2 \) is not always valid. However, the above identity is true iff \( \arg z_1 + \arg z_2 \in (-\pi, \pi] \) (why?).

In calculus, interesting examples of differentiable functions, apart from polynomials and exponential, are given by trigonometric functions. The situation is similar for functions of complex variables.

If \( x \in \mathbb{R} \) then using Taylor series for sine and cosine we get
\[
e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \cos x + i \sin x.
\]

Taking clue from the above, we now define
\[
\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.
\]

It is easy to see by ratio test that the radius of convergence of these two power series is \( \infty \). It now follows easily that \( e^{iz} = \cos z + i \sin z \) (Euler’s formula) and hence
\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.
\]

Using the above formulae the following theorem follows easily.

**Theorem 3.** For any \( z \in \mathbb{C} \)

1. \( \sin(-z) = -\sin z, \cos(-z) = \cos z, \sin(z+2k\pi) = \sin z, \cos(z+2k\pi) = \cos z, \sin^2 z + \cos^2 z = 1. \)
2. \( \frac{d}{dz}(\sin z) = \cos z, \frac{d}{dz}(\cos z) = -\sin z. \)
3. \( \sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \) and \( \cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y \) where \( \sinh x = \frac{e^x - e^{-x}}{2}, \cos x = \frac{e^x + e^{-x}}{2}. \)

There is an important difference between real and complex sine functions. Unlike the real sine function the complex sine function is unbounded. To see this notice that \( |\sin z|^2 = |\sin(x + iy)|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x + \sinh^2 y. \)

As \( \lim_{y \to \infty} \sinh y = \infty \) (check this!) it follows that for each fixed \( x_0 \in \mathbb{R} \), \( \lim_{y \to \infty} |\sin(x_0 + iy)| = \infty. \) Similar is the case for \( \cos z. \)

Now using sine and cosine we can define \( \tan z, \sec z, \cosec z \) as in the real case. We can also define complex analogue of the hyperbolic functions \( \sinh z = (e^z - e^{-z})/2 \) and \( \cosh z = (e^z + e^{-z})/2 \).
and \( \cosh z = (e^z + e^{-z})/2 \). The following theorem follows just by applying the definitions

**Theorem 4.**

1. \( \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \), \( \sin 2z = 2 \sin z \cos z \),
   \( \sin(z + \pi) = -\sin z \), \( \sin(z + 2\pi) = \sin z \).

2. \( \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \), \( \cos 2z = \cos^2 z - \sin^2 z \).