LECTURE 7: CAUCHY'S THEOREM

The analogue of the fundamental theorem of calculus proved in the last lecture says in particular that *if a continuous function f has an antiderivative F in a domain D, then $\int_C f(z)dz = 0$ for any given closed contour lying entirely on D.*

Now, two questions arises: 1) Under what conditions on f we can guarantee the existence of F such that $F' = f$? 2) Under what assumptions on f, we can get $\int_c f(z)dz = 0$ for a closed contour?

Cauchy's theorem answers the questions raised above. To state Cauchy's theorem we need some new concepts.

Definition 1. *(Simply connected domain)*
A domain D is called simply connected if every simple closed contour (within it) encloses points of D only.

A domain D is called multiply connected if it is not simply connected.

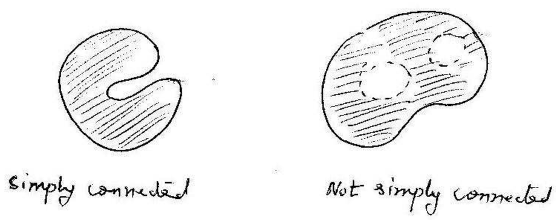

FIGURE 1

Example 2. *Here are some examples:*

(1) *The sets $\mathbb{C}$, $\mathbb{D}$, and $RHP = \{z : Re\ z > 0\}$ are simply connected domains (they have no holes).*
(2) *The sets $\mathbb{C}^*$, $\mathbb{D} \setminus \{0\}$, and the annulus $A(a,b) = \{z \in \mathbb{C} : a < |z| < b\}$ are not simply connected domains.*

Definition 3. *A curve (contour) is called simple if it does not cross itself (if initial point and the final point are same they are not considered as non simple)*

A curve is called a simple closed curve if the curve is simple and its initial point and final point are same.

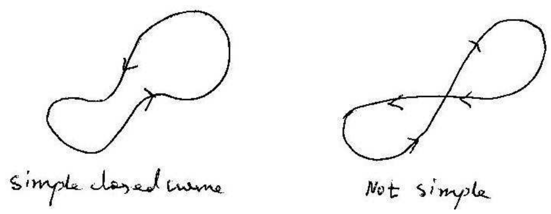

FIGURE 2

Example 4. *For* $z_0 \in \mathbb{C}$ *and* $r > 0$ *the curve* $\gamma(z_0, r)$ *given by the function* $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi)$ *is a prototype of a simple closed curve (which is the circle around* z_0 *with radius* r*).*

Theorem 5. *If a function* f *is analytic on a simply connected domain* D *and* C *is a simple closed contour lying in* D *then* $\int_C f(z)dz = 0$.

We will prove the theorem under an extra hypothesis that f' is a continuous function.

Proof. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $\gamma(t) = x(t) + iy(t)$, $a \le t \le b$ is the curve C. Then

$$\begin{aligned}
\int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b [u(x(t),y(t)) + iv(x(t),y(t))][x'(t) + iy'(t)]dt \\
&= \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt \\
&= \int_C udx - vdy + i\int_C vdx + udy \\
&= \int\int_R (-v_x - u_y)dxdy + i\int\int_R (u_x - v_y)dxdy, \\
&\qquad \text{(by Green's theorem)} \\
&= 0 \qquad \text{(by CR equations).}
\end{aligned}$$

□

At this point we pause a bit and take a stock of the method of evaluation of integrals:

(1) We can straightway use the parametrization of the curve and apply the definition, as we did for the evaluation of the *fundamental integral.* But this method can turn out to be tedious.

(2) We can recognize the integrand as a continuous derivative of another function and apply the analogue of the fundamental theorem.
(3) If all the conditions are met we can use Cauchy's theorem.

Example 6. *Let $\gamma(t) = e^{it}$, $-\pi < i \leq \pi$, and C denotes the circle of radius one with center at zero.*

(1) *It follows from Cauchy's theorem that $\int_C f(z)dz = 0$, if $f(z) = e^{z^n}$, $\cos z$, or $\sin z$.*
(2) *$\int_C f(z)dz = 0$ if $f(z) = \frac{1}{z^2}$, or $cosec^2 z$ from the fundamental theorem as $\frac{d}{dz}(-\frac{1}{z}) = \frac{1}{z^2}$ and $\frac{d}{dz}(-\cot z) = cosec^2 z$. Note that here Cauchy's theorem cannot be applied as the integrands are not analytic at zero.*
(3) *$\int_C \frac{e^{iz^2}}{z^2+4}dz = 0$ by Cauchy's theorem. Note that the integrand is not analytic at $z = \pm 2$ but that does not bother us as these points are not enclosed by C.*
(4) *If $f(z) = (Im\ z)^2$ then $\int_C f(z)dz = 0$ (**check this**). As f is not analytic anywhere in $\mathbb{C}$ Cauchy's theorem can not be applied to prove this.*

Important consequences: We have the following important consequences of Cauchy's theorem.

(1) (*Independence of path*) Let D be a simply connected domain and f be an analytic function defined on D. Let z_1, z_2 be two points in D and γ_1 and γ_2 be two simple curves joining z_1 and z_2 such that the curves lie entirely in D. Then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$. To see this consider the curve $\gamma(t) = \gamma_1(2t)$, $0 \leq t \leq 1/2$ and $\gamma(t) = \eta(t) = \gamma_2(2(1-t))$ for $1/2 \leq t \leq 1$ (we have just reversed the direction of γ_2 and joined it with γ_1). Then γ is a simple closed curve and by Cauchy's theorem $\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_\eta f(z)dz = 0$ which implies $\int_{\gamma_1} f(z)dz = -\int_\eta f(z)dz$. But as $-\int_\eta f(z)dz = \int_{\gamma_2} f(z)dz$ we get the result.
(2) (*Existence of antiderivative:*) If f is an analytic function on a simply connected domain D then there exists a function F, which is analytic on D such that $F' = f$.

Proof. (*) Fix a point $z_0 \in D$ and define

$$F(z) = \int_{z_0}^{z} f(w)dw.$$

The integral is considered as a contour integral over any curve lying in D and joining z with z_0. By the first part the integral does not depend on the curve we choose and hence the function F is well defined. We will show that

$F' = f$. If $z + h \in D$ then

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)dw - \int_{z_0}^{z} f(w)dw = \int_{z}^{z+h} f(w)dw,$$

where the curve joining z and $z + h$ can be considered as a straight line $l(t) = z + th$, $t \in [0, 1]$. Thus we get

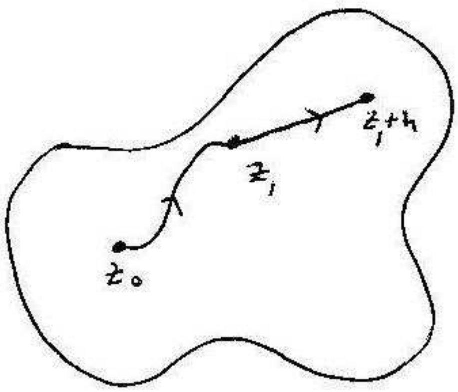

FIGURE 3

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| = \frac{1}{h}\left|\int_{z}^{z+h} (f(w) - f(z))dw\right|,$$

(here we have used the fact that $\int_l dw = h$). Since f is continuous at z, given $\epsilon > 0$ there exist a $\delta > 0$ such that $|f(z+h) - f(z)| < \epsilon$ if $|h| < \delta$. Thus for $|h| < \delta$ we get from ML inequality that

$$\frac{1}{h}\left|\int_{z}^{z+h} (f(w) - f(z))dw\right| \leq \frac{\epsilon|h|}{|h|},$$

that is, $\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$.

□

Cauchy's theorem for multiply connected domain: See the discussion in Page 719 of *Advanced Engineering Mathematics-E. Kreyszig*