LECTURE 9: CAUCHY’S INTEGRAL FORMULA II

Let us first summarize Cauchy’s theorem and Cauchy’s integral formula. Let $C$ be a simple closed curve contained in a simply connected domain $D$ and $f$ is an analytic function defined on $D$. Then

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz = \begin{cases} 
2\pi if(z_0), & \text{if } n = 0 \text{ and } z_0 \text{ is enclosed by } C. \\
\frac{2\pi i}{n!}f^n(z_0), & \text{if } n \geq 1 \text{ and } z_0 \text{ is enclosed by } C. \\
0, & \text{if } z_0 \text{ lies outside the region enclosed by } C.
\end{cases}$$

By Cauchy’s integral formula one can also tackle integrals of the form $$\int_C \frac{f(z)}{(z - z_0)(z - z_1)}dz$$ where the simple closed curve $C$ includes two points $z_0, z_1$. By using partial fraction we get that

$$\int_C \frac{f(z)}{(z - z_0)(z - z_1)}dz = \int_C \frac{f(z)}{z_0 - z_1}(\frac{1}{z - z_0} - \frac{1}{z - z_1})dz = \frac{2\pi i(\frac{f(z_0) - f(z_1)}{(z_0 - z_1)})}{(z_0 - z_1)}.$$

Example 1. If $a \in \mathbb{C}$ then

$$\int_{\{z:|z|=2\}} \frac{e^{az}}{z^2 + 1}dz = \int_{\{z:|z|=2\}} \frac{e^{az}}{(z + i)(z - i)}dz = \frac{e^{-ia} - e^{ia}}{4\pi}.$$

We will now see some more serious application of CIF. For $r > 0$ let us define $B_r(z_0) = \{z : |z - z_0| \leq r\}$ and $S_r(z_0) = \{z : |z - z_0| = r\}$.

Theorem 2. (Cauchy’s estimate) Suppose that $f$ is analytic on a simply connected domain $D$ and $\overline{B_R(z_0)} \subset D$ for some $R > 0$. If $|f(z)| \leq M$ for all $z \in S_R(z_0)$, then for all $n \geq 0$,

$$|f^n(z_0)| \leq \frac{n!M}{R^n}.$$

Proof. From Cauchy’s integral formula and $ML$ inequality we have

$$|f^n(z_0)| = |\frac{n!}{2\pi i} \int_{\partial S_R(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}dz| \leq \frac{n!}{2\pi} M \frac{1}{R^{n+1}} 2\pi R = \frac{n!M}{R^n}.$$

As a consequence of the above theorem we get the following miraculous result.

Theorem 3. (Liouville’s Theorem) If $f$ is analytic and bounded on the whole $\mathbb{C}$ then $f$ is a constant function.
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Proof. To prove this we will prove that \( f' \) is the zero function. Choose \( \epsilon > 0 \) arbitrary and choose any point \( z_0 \in \mathbb{C} \). Now consider \( B_R(z_0) \) such that \( R > M/\epsilon \) (for small \( \epsilon \), \( R \) will be very large but that is not a problem as \( f \) is analytic everywhere). By Cauchy’s estimate now we have,

\[
|f'(z_0)| \leq \frac{M}{R} < \epsilon.
\]

Hence \( f'(z_0) = 0 \). But \( z_0 \) is arbitrary and hence \( f'(z) = 0 \) for all \( z \in \mathbb{C} \). \( \Box \)

Remark: We have earlier observed that \( \cos z \) and \( \sin z \) are not bounded in \( \mathbb{C} \). Another proof of the same fact now follows from Liouville’s theorem. Moreover it shows that this behavior is typical of non constant analytic functions on \( \mathbb{C} \). Thus if a function is bounded it cannot be analytic on whole \( \mathbb{C} \).

We now show another application of Liouville’s theorem to prove the Fundamental Theorem of Algebra.

**Theorem 4.** Every polynomial \( p(z) \) of degree \( n \geq 1 \) has a root (in \( \mathbb{C} \)).

Proof. Suppose \( P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0 \) is a polynomial with no root in \( \mathbb{C} \). Then \( \frac{1}{P(z)} \) is analytic on whole \( \mathbb{C} \). Since

\[
\left| \frac{P(z)}{z^n} \right| = \left| 1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| \to 1, \text{ as } |z| \to \infty,
\]

it follows that \( |p(z)| \to \infty \) and hence \( |1/p(z)| \to 0 \) as \( |z| \to \infty \) (we are just proving a well known fact that polynomials are unbounded functions). Consequently \( \frac{1}{p(z)} \) is a bounded function. Hence by Liouville’s theorem \( \frac{1}{p(z)} \) is constant which is impossible. \( \Box \)

We will now prove a partial converse to Cauchy’s theorem

**Theorem 5.** (Morera’s theorem) If \( f \) is continuous in a simply connected domain \( D \) and if \( \int_C f(z) dz = 0 \) for every simple closed contour \( C \) in \( D \) then \( f \) is analytic

Proof. The idea is just to prove that there exists an analytic function \( F \) such that \( F' = f \). Then we can use CIF to conclude that \( f \) is analytic. So, fix a point \( z_0 \in D \) and define \( F(z) = \int_{z_0}^z f(w)dw \) (by hypothesis it does not matter which closed curve I use). By using continuity, we can show as before that \( F \) is analytic and \( F'' = f \). \( \Box \)

The next theorem shows that an analytic function is always given by a power series.
Theorem 6. (Taylor’s Theorem)
Let \( f \) be analytic on \( D = \{ z : |z - z_0| < R_0 \} \). Then
\[
f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad \text{for all } z \in D,
\]
where \( a_n = \frac{f^{(n)}(z_0)}{n!} \) for \( n = 0, 1, 2, \ldots \).

Proof. (*) Without loss of generality we consider \( z_0 = 0 \). Fix \( z \in D \). Let \( |z| = r \) and \( C_0 \) be a circle with center 0 and radius \( r_0 \) such that \( r < r_0 < R_0 \). We need the following algebraic identity,
\[
\frac{1}{1-q} = 1 + q + q^2 + \ldots + q^{n-1} + \frac{q^n}{1-q},
\]
which follows easily from
\[
1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1-q}.
\]
Thus for two complex numbers \( w \) and \( z \) we can write
\[
(0.1) \quad \frac{1}{w-z} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \ldots + \frac{z^{n-1}}{w^n} + \frac{z^n}{(w-z)w^n}.
\]
By CIF and (0.1) we now have
\[
f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)dw}{w-z}
= \frac{1}{2\pi i} \int_{C_0} f(w) \left[ \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \ldots + \frac{z^{n-1}}{w^n} \right] dw + \frac{z^n}{2\pi i} \int_{C_0} \frac{f(w)dw}{(w-z)w^n}
= f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!} z^{n-1} + \rho_n(z)
\]
where \( \rho_n(z) = \frac{z^n}{2\pi i} \int_{C_0} \frac{f(w)dw}{(w-z)w^n} \). Now, we just need to show that \( \lim_{n \to \infty} |\rho_n(z)| = 0 \).
Notice that the function \( w \to \frac{f(w)}{w-z} \) is a bounded function on the circle \( C_0 \) (as it is continuous). Thus by \( ML \) inequality it follows that
\[
|\rho_n(z)| \leq K r_0 \left| \frac{z}{r_0} \right|^n.
\]
As \( |z| = r < r_0 \) it follows that the right hand side goes to zero as \( n \to \infty \). \( \square \)